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FIELDS ON THE POINCARÉ GROUP: Arbitrary Spin Description and Relativistic Wave Equations

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In this paper, starting from pure group-theoretical point of view, we develop a regular approach to describing particles with different spins in the framework of a theory of scalar fields on the Poincaré group. Such fields can be considered as generating functions for conventional spin-tensor fields. The cases of 2, 3, and 4 dimensions are elaborated in detail. Discrete transformations C, P, T are defined for the scalar fields as automorphisms of the Poincaré group. Doing a classification of the scalar functions, we obtain relativistic wave equations for particles with definite spin and mass. There exist two different types of scalar functions (which describe the same mass and spin), one related to a finite-dimensional nonunitary representation and another one related to an infinite-dimensional unitary representation of the Lorentz subgroup. This allows us to derive both usual finite-component wave equations for spin-tensor fields and positive energy infinite-component wave equations.

I. INTRODUCTION

Traditionally in field theory particles with different spins are described by multicomponent spin-tensor fields on Minkowski space. However, it is possible to use for this purpose scalar functions as well, which depend on both Minkowski space coordinates and on some continuous bosonic variables corresponding to spin degrees of freedom. For the first time, such fields were introduced in [1–4] in connection with the problem of constructing relativistic wave equations (RWE). Fields of this type may be treated as ones on homogeneous spaces of the Poincaré group. A systematic development of such point of view was given by Finkelstein [5]. He also gave a classification and explicit constructions of homogeneous spaces of the Poincaré group, which contain Minkowski space. The next logical step was done by Lurçat [6] who suggested to construct quantum field theory on the Poincaré group. One of the motivations was to give a dynamical role to the spin. Some development of these ideas was given in [7–13]. For example, different homogeneous spaces were described, as well as possibilities to introduce interactions in spin phase space, and to construct Lagrangian formulations were studied. The authors of [7] arrived at the conclusion that eight is the lowest dimension of a homogeneous space suitable for a description of both half-integer

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and integer spins. However, no convinced physical motivation for the choice of homogeneous spaces was presented, and the interpretation of additional degrees of freedom and of corresponding quantum numbers remained an open problem.

In this paper, starting from pure group-theoretical point of view, we develop a regular approach to describing particles with different spins in the framework of a theory of scalar fields on the Poincaré group. Such fields can be considered as generating functions for conventional spin-tensor fields. In this language the problem of constructing RWE of different types is formulated from a unique position.

In our consideration, we use scalar fields on the proper Poincaré group, i.e., fields on the ten-dimensional manifold; this manifold is a direct product of Minkowski space and of the manifold of the Lorentz subgroup. These fields arise in our constructions in course of the study of a generalized regular representation (GRR). That provides a possibility to analyze then all the representations of the Poincare group. The study of GRR implies a wide use of harmonic analysis method [14–17]. In a sense, this method is an alternative to one of induced representations suggested by Wigner [18] (see [16,19–21]). It turns out that the fields on the Poincaré group can be considered as generating functions for usual spin-tensor fields on Minkowski space, thus we naturally obtain all results for the latter fields. However, sometimes it is more convenient to formulate properties and equations for spin-tensor fields in terms of the generating functions. Moreover, the problem of constructing RWE looks very natural in the language of the scalar fields on the group. We show that this problem can be formulated as a problem of a classification of different scalar fields. For this purpose, in accordance with the general theory of harmonic analysis, we consider various sets of commuting operators and identify constructing RWE with eigenvalue problems for this operators. We succeeded to define discrete transformations for the scalar fields using some automorphisms of the proper Poincare group. The space of scalar fields on the group turns out to be closed with respect to the discrete transformations. One ought to say that the latter transformations are of fundamental importance for constructing RWE and for their analysis. Consideration of the discrete transformations helps us to give right physical interpretation for quantum numbers which appear in course of the classification of the scalar fields.

The paper is organized as follows.

In Sect. 2 we introduce the basic objects of our study, namely, scalar fields $f(x, \mathbf{z})$. The scalar fields depend on x, which are coordinates on Minkowski space, and on $\mathbf{z}$, which are coordinates on the Lorentz subgroup. The complex coordinates $\mathbf{z}$ describe spin degrees of freedom. It is shown that these fields are generating functions for usual spins-tensor fields. Classifying the scalar fields with the help of various sets of commuting operators on the group, we get description of irreps of the group. We formulate a general scheme of constructing RWE in this language in any dimensions. We introduce discrete transformations in the space of the scalar functions and we relate these transformations to automorphisms of the proper Poincaré group.

In Sect. 3 we apply the above general scheme to detailed study of scalar fields on two-dimensional Poincaré and Euclidean groups. In particular, we construct RWE and analyze their solutions.

Three-dimensional Poincaré and Euclidean group case is considered in Sect. 4. Besides finite-component equations, we also construct positive energy RWE assotiated with unitary

infinite-dimensional irreps of 2+1 Lorentz group. These equations, in particular, describe particles with fractional spins.

In Sect. 5 we study scalar fields on the $3+1$ proper Poincaré group. A connection of the present consideration with other approaches to RWE theory is considered in detail. In particular, we pay significant attention to equations with subsidiary conditions. General first-order Gel'fand–Yaglom equations (including Bhabha equations), Dirac–Fierz–Pauli equations, and Rarita–Schwinger equations arise in the present consideration as well. This give a regular base for comparison of properties of various RWE.

Doing the classification of scalar functions in 2, 3, and 4 dimensions, we obtain equations describing fields with fixed mass and spin. In Sect. 6 we consider the general features of these equations.

One ought to say that the construction of RWE is elaborated in detail only for the massive case. We plane to discuss the massless case in a later article.

II. FIELDS ON THE PROPER POINCARÉ GROUP AND SPIN DESCRIPTION

A. Parametrization of the Poincaré group

Consider Poincaré group transformations

$$x'^{\nu} = \Lambda^{\nu}{}_{\mu} x^{\mu} + a^{\nu} \tag{2.1}$$

of coordinates $x = (x^\mu,\ \mu = 0, \ldots, D)$ in $d = D+1$-dimensional Minkowski space, $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$, $\eta_{\mu\nu} = \mathrm{diag}(1, -1, \ldots, -1)$. The matrices Λ define rotations in Minkovski space and belong to the vector representation of $O(D,1)$ group. We are also going to consider D-dimensional Euclidean case in which $ds^2 = \eta_{ik} dx^i dx^k$, and $\eta_{ik} = \mathrm{diag}(1, 1, \ldots, 1)$, $i, k = 1, \ldots, D$. Here the matrices Λ belong to the vector representation of $O(D)$ group.

The transformations (2.1) which can be obtained continuously from the identity form the proper Poincaré group $M_0(D,1)$ with the elements $g = (a, \Lambda)$. Corresponding homogeneous transformations ($a = 0$) form the proper Lorentz group $SO_0(D,1)$. In the Euclidean case we deal with $M_0(D)$ and $SO(D)$ respectively. The composition law and the inverse element of these groups have the form

$$(a_2, \Lambda_2)(a_1, \Lambda_1) = (a_2 + \Lambda_2 a_1, \Lambda_2 \Lambda_1), \quad g^{-1} = (-\Lambda^{-1} a, \Lambda^{-1}). \tag{2.2}$$

Thus, the groups $M_0(D,1)$ and $M_0(D)$ are semidirect products

$$M_0(D,1) = T(d) \times) SO_0(D,1), \qquad M_0(D) = T(D) \times) SO_(D),$$

where T(d) is d-dimensional translation group.

There exists one-to-one correspondence between the vectors x and 2×2 Hermitian matrices X in pseudo-Euclidean spaces of 2,3 and 4 dimensions,[1]

[1] We use two sets of 2×2 matrices $\sigma_\mu = (\sigma_0, \sigma_k)$ and $\bar{\sigma}_\mu = (\sigma_0, -\sigma_k)$,

$$x \leftrightarrow X, \quad X = x^\mu \sigma_\mu. \tag{2.3}$$

Namely:

$$d = 3+1: \quad X = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}, \tag{2.4}$$

$$d = 2+1: \quad X = \begin{pmatrix} x^0 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 \end{pmatrix}, \tag{2.5}$$

$$d = 1+1: \quad X = \begin{pmatrix} x^0 & x^1 \\ x^1 & x^0 \end{pmatrix}. \tag{2.6}$$

In all the above cases

$$\det X = \eta_{\mu\nu} x^\mu x^\nu, \quad x^\mu = \frac{1}{2} \operatorname{Tr}(X \bar{\sigma}^\mu). \tag{2.7}$$

In Euclidean spaces of 2 and 3 dimensions a similar correspondence has the form

$$D = 3: \qquad X = \begin{pmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{pmatrix}, \tag{2.8}$$

$$D = 2: \qquad X = \begin{pmatrix} x^2 & x^1 \\ x^1 & -x^2 \end{pmatrix}. \tag{2.9}$$

If x is subjected to a transformation (2.1), then X transforms as follows (see, for example, [14]:

$$X' = UXU^\dagger + A, \tag{2.10}$$

where $A = a^\mu \sigma_\mu$ and U are some 2×2 complex matrices obeying the conditions

$$\sigma_\nu \Lambda^\nu{}_\mu = U \sigma_\mu U^\dagger. \tag{2.11}$$

Eq. (2.11) relates the matrices Λ and U. There are many U which correspond to the same Λ. We may fix this arbitrariness imposing the condition

$$\det U = 1, \tag{2.12}$$

which does not contradict to the relation $\det U = e^{i\phi}$, which follows from (2.11). However, even after that, there is no one-to-one correspondence between Λ and U, namely two matrices $(U, -U)$ correspond to one Λ. Considering both U and $-U$ as representatives for Λ, we in fact go over from $SO_0(D,1)$ to its double covering group $\mathrm{Spin}(D,1)$, or, in the Euclidean case, from $SO(D)$ to its double covering group $\mathrm{Spin}(D)$. In the dimensions under consideration the groups $\mathrm{Spin}(D,1)$ and $\mathrm{Spin}(D)$ are isomorphic to the following ones:[2]

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

[2] We denote the complex conjugation by $*$ above the quantities.

$$d=3+1: \quad U \in SL(2,C), \quad U=\begin{pmatrix} u_1^1 & u_2^1 \\ u_1^2 & u_2^2 \end{pmatrix}, \quad u_1^1 u_2^2 - u_1^2 u_2^1 = 1, \tag{2.13}$$

$$d=2+1: \quad U \in SU(1,1), \quad U=\begin{pmatrix} u_1 & u_2 \\ \overset{*}{u}_2 & \overset{*}{u}_1 \end{pmatrix}, \quad |u_1|^2 - |u_2|^2 = 1, \tag{2.14}$$

$$D=3: \quad U \in SU(2), \quad U=\begin{pmatrix} u_1 & u_2 \\ -\overset{*}{u}_2 & \overset{*}{u}_1 \end{pmatrix}, \quad |u_1|^2 + |u_2|^2 = 1, \tag{2.15}$$

$$d=1+1: \quad U \in SO(1,1), \quad U=\begin{pmatrix} \cosh\frac{\phi}{2} & \sinh\frac{\phi}{2} \\ \sinh\frac{\phi}{2} & \cosh\frac{\phi}{2} \end{pmatrix}, \tag{2.16}$$

$$D=2: \quad U \in SO(2), \quad U=\begin{pmatrix} \cos\frac{\phi}{2} & \sin\frac{\phi}{2} \\ -\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix}. \tag{2.17}$$

Considering nonhomogeneous transformations and retaining both elements U and $-U$ in the consideration, we go over from the groups $M_0(D,1)$ and $M_0(D)$ to the groups

$$M(D,1) = T(d)\times)\mathrm{Spin}(D,1), \qquad M(D) = T(D)\times)\mathrm{Spin}(D)$$

respectively. As it is known, that allows one to avoid double-valued representations for half integer spins. Thus, there exists one-to-one correspondence between the elements g of the groups $M(D,1)$, $M(D)$ and two 2×2 matrices, $g \leftrightarrow (A,U)$. The first one A corresponds to translations and the second one U corresponds to rotations. Eq. (2.10) describes the action of $M(D,1)$ on Minkowski space (the latter is coset space $M(D,1)/\mathrm{Spin}(D,1)$). As a consequence of (2.10), one can obtain the composition law and the inverse element of the groups $M(D,1)$, $M(D)$:

$$(A_2,U_2)(A_1,U_1) = (U_2 A_1 U_2^\dagger + A_2,\ U_2 U_1)\,, \quad g^{-1} = (-U^{-1}A(U^{-1})^\dagger,\ U^{-1})\,. \tag{2.18}$$

The matrices U in the dimensions under consideration satisfy the following identities:

$$U \in SL(2,C): \quad \sigma_2 U \sigma_2 = (U^T)^{-1}; \tag{2.19}$$

$$U \in SU(1,1): \quad \sigma_1 U \sigma_1 = \overset{*}{U}, \quad \sigma_2 U \sigma_2 = (U^T)^{-1}, \quad \sigma_3 U \sigma_3 = (U^\dagger)^{-1}, \tag{2.20}$$

$$U \in SU(2): \quad \sigma_2 U \sigma_2 = (U^T)^{-1} = \overset{*}{U}. \tag{2.21}$$

An equivalent picture arise in terms of the matrices $\overline{X} = x^\mu \bar\sigma_\mu$. Using the relation $\overline{X} = \sigma_2 X^T \sigma_2$, the transformation law for X (2.10), and the identity (2.19), one can get

$$\overline{X}' = (U^\dagger)^{-1}\overline{X}U^{-1} + \overline{A}. \tag{2.22}$$

Thus, $\overline{X}$ are transformed by means of the elements $(\overline{A},(U^\dagger)^{-1})$. The relation $(A,U) \to (\overline{A},(U^\dagger)^{-1})$ defines an automorphism of the Poincaré group $M(D,1)$. In Euclidean case the matrices U are unitary, and the latter relation is reduced to $(A,U) \to (-A,U)$.

The representation of the Poincaré transformations in the form (2.10) is closely related to a representation of finite rotations in $\mathbb{R}^d$ in terms of the Clifford algebra. In higher dimensions the transformation law has the same form, where A is a vector element and U corresponds to an invertible element (spinor element) of the Clifford algebra [22]. Besides, the representation of the finite transformations in the form (2.10) can be useful for spin description by means of Grassmannian variables ξ, since ξ and $\partial\xi$ give a realization of the Clifford algebra [23].

B. Regular representation and scalar functions on the group

It is well known [17,14,15] that any irrep of a group G is contained (up to the equivalence) in a decomposition of a GRR. Thus, the study of GRR is an effective method for the analysis of irreps of the group. Consider, first, the left GRR $T_L(g)$, which is defined in the space of functions $f(g_0)$, $g_0 \in G$, on the group, as

$$T_L(g)f(g_0) = f'(g_0) = f(g^{-1}g_0),\ g \in G. \tag{2.23}$$

As a consequence of the relation (2.23), we can write

$$f'(g_0') = f(g_0), \quad g_0' = gg_0. \tag{2.24}$$

Let G be the group $M(3,1)$, and we use the parametrization of its elements by two 2×2 matrices (one hermitian and another one from $SL(2,C)$), which was described in the previous Sect. At the same time, using such a parametrization, we choose the following notations:

$$g \leftrightarrow (A,U)\,,\ g_0 \leftrightarrow (X,Z)\,, \tag{2.25}$$

where A, X are 2×2 hermitian matrices and $U, Z \in SL(2,C)$. The map $g_0 \leftrightarrow (X,Z)$ creates the correspondence

$$\begin{aligned} &g_0 \leftrightarrow (x, z, \underline{z}), \quad \text{where} \quad x = (x^\mu),\ z = (z_\alpha),\ \underline{z} = (\underline{z}_\alpha),\\ &\mu = 0,1,2,3,\ \alpha = 1,2,\ z_1\underline{z}_2 - z_2\underline{z}_1 = 1, \end{aligned} \tag{2.26}$$

by virtue of the relations

$$X = x^\mu\sigma_\mu, \quad Z = \begin{pmatrix} z_1 & \underline{z}_1 \\ z_2 & \underline{z}_2 \end{pmatrix} \in SL(2,C)\,. \tag{2.27}$$

On the other hand, we have the correspondence $g_0' \leftrightarrow (x', z', \underline{z}')$,

$$\begin{aligned} &g_0' = gg_0 \leftrightarrow (X', Z') = (A,U)(X,Z) = (UXU^+ + A, UZ) \leftrightarrow (x', z', \underline{z}'),\\ &x'^\mu\sigma_\mu = X' = UXU^+ + A \quad \Longrightarrow \quad x'^\mu = (\Lambda_0)^\mu{}_\nu x^\nu + a^\mu, \quad \Lambda \leftarrow U \in SL(2,C), \end{aligned} \tag{2.28}$$

$$\begin{pmatrix} z_1' & \underline{z}_1' \\ z_2' & \underline{z}_2' \end{pmatrix} = Z' = UZ \implies z_\alpha' = U_\alpha{}^\beta z_\beta,\ \underline{z}_\alpha' = U_\alpha{}^\beta \underline{z}_\beta,\ U = (U_\alpha{}^\beta),\ z_1'\underline{z}_2' - z_2'\underline{z}_1' = 1. \tag{2.29}$$

Then the relation (2.24) takes the form

$$f'(x', z', \underline{z}') = f(x, z, \underline{z}), \tag{2.30}$$

$$x'^\mu = (\Lambda_0)^\mu{}_\nu x^\nu + a^\mu, \quad \Lambda \leftarrow U \in SL(2,C), \tag{2.31}$$

$$z_\alpha' = U_\alpha{}^\beta z_\beta, \quad \underline{z}_\alpha' = U_\alpha{}^\beta \underline{z}_\beta, \quad z_1\underline{z}_2 - z_2\underline{z}_1 = z_1'\underline{z}_2' - z_2'\underline{z}_1' = 1. \tag{2.32}$$

The relations (2.30)-(2.32) admit a remarkable interpretation. We may treat x and x' in these relations as position coordinates in Minkowski space (in different Lorentz refrence frames) related by proper Poincare transformations, and the sets $(z, \underline{z})$ and $(z, \underline{z}')$ may be treated as spin coordinates in these Lorentz frames. They are transformed according to the formulas (2.32). Carrying two-dimensional spinor representation of the Lorentz group,

the variables z and $\underline{z}$ are invariant under translations as one can expect for spin degrees of freedom. Thus, we may treat sets $(x, z, \underline{z})$ as points in a position-spin space with the transformation law (2.31), (2.32) under the change from one Lorentz reference frame to another. In this case equations (2.30)-(2.32) present the transformation law for scalar functions on the position-spin space.

On the other hand, as we have seen, the sets $(x, z, \underline{z})$ are in one-to-one correspondence to the group $M(3,1)$ elements. Thus, the functions $f(x, z, \underline{z})$ are still functions on this group. That is why we often call them scalar functions on the group as well, remembering that the term "scalar" came from the above interpretation.

Remember now that different functions of such type correspond to different representations of the group $M(3,1)$. Thus, the problem of classification of all irreps of this group is reduced to the problem of a classification of all scalar functions on position-spin space. However, for the purposes of such classification, it is natural to restrict ourselves by the scalar functions which are analytic both in $z, \underline{z}$ and in $\overset{*}{z}, \overset{*}{\underline{z}}$ (or, simply speaking, which are differentiable with respect to these arguments). Further such functions are denoted by $f(x, z, \underline{z}, \overset{*}{z}, \overset{*}{\underline{z}}) = f(x, \mathbf{z})$, $\mathbf{z} = (z, \underline{z}, \overset{*}{z}, \overset{*}{\underline{z}})$.

Consider now the right GRR $T_R(g)$. This representation is defined in the space of functions $f(g_0)$, $g_0 \in G$ as

$$T_R(g) f(g_0) = f'(g_0) = f(g_0 g), \quad g \in G, \tag{2.33}$$

As a consequence of the relation (2.33), we can write

$$f'(g_0') = f(g_0), \quad g_0' = g_0 g^{-1}. \tag{2.34}$$

In the case of the proper Poincare group, the right transformations act on $g_0 \leftrightarrow (X, Z)$ according to the formula

$$g_0' = g_0 g^{-1} \leftrightarrow (X', Z') = (X + Z^{-1} A (Z^{-1})^\dagger, Z U^{-1}), \tag{2.35}$$

hence $x'^\mu = x^\mu + L^\mu{}_\nu a^\nu$, where the matrix L depends on $\mathbf{z}$, $\sigma_\nu L^\nu{}_\mu = Z^{-1} \sigma_\mu (Z^{-1})^\dagger$. The transformations for $x, \mathbf{z}$ do not admit similar to the left GRR case interpretation. In particular, the transformation low for x does not look as a Lorentz transformation. On the other hand, the study of the right GRR is useful for the purposes of the classification of the Poincare group irreps, since the generators of the right GRR are used to construct complete sets of commuting operators on the group.

C. Generators of generalized regular representations

Generators of the left GRR correspond to translations and rotations. They can be written as

$$\hat{p}_\mu = -i\partial/\partial x^\mu, \quad \hat{J}_{\mu\nu} = \hat{L}_{\mu\nu} + \hat{S}_{\mu\nu}, \tag{2.36}$$

where $\hat{L}_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$ are angular momentum operators, and $\hat{S}_{\mu\nu}$ are spin operators depending on $\mathbf{z}$ and $\partial/\partial\mathbf{z}$. An explicit form of spin operators is given in the Appendix.

The algebra of the generators (2.36) has the form

$$
\begin{aligned}
&[\hat{p}_\mu, \hat{p}_\nu] = 0, \quad [\hat{J}_{\mu\nu}, \hat{p}_\rho] = i(\eta_{\nu\rho}\hat{p}_\mu - \eta_{\mu\rho}\hat{p}_\nu), \\
&[\hat{J}_{\mu\nu}, \hat{J}_{\rho\sigma}] = i\eta_{\nu\rho}\hat{J}_{\mu\sigma} - i\eta_{\mu\rho}\hat{J}_{\nu\sigma} - i\eta_{\nu\sigma}\hat{J}_{\mu\rho} + i\eta_{\mu\sigma}\hat{J}_{\nu\rho}\ .
\end{aligned}
\tag{2.37}
$$

In the space of Fourier transforms

$$
\varphi(p, \mathbf{z}) = (2\pi)^{-d/2} \int f(x, \mathbf{z}) e^{ipx} dx \tag{2.38}
$$

the left GRR acts as (one has to use (2.23)):

$$
T_L(g)\varphi(p, \mathbf{z}) = e^{iap'}\varphi(p', g^{-1}\mathbf{z}), \quad p' = g^{-1}p \leftrightarrow P' = U^{-1}P(U^{-1})^\dagger, \quad P = p_\mu\sigma^\mu. \tag{2.39}
$$

One can see that $\det Z$ and $\det P = p^2$ are invariant under the transformations[3] (2.39) and that p^2 is an eigenvalue of the Casimir operator $\hat{p}^2$.

For the groups $M(D)$ there are two types of representations depending on p^2: 1) $p^2 \neq 0$; 2) $p^2 = 0$; then all $p_i = 0$, and irreps are labelled by eigenvalues of Casimir operators of the rotation subgroup.

For the groups $M(D,1)$ there are four types of representations depending on the eigenvalues m^2 of the Casimir operator $\hat{p}^2$: 1) $m^2 > 0$; 2) $m^2 < 0$ (tachyon); 3) $m^2 = 0$, $p_0 \neq 0$ (massless particle); 4) $m^2 = p_0 = 0$, irreps are labelled by eigenvalues of the Casimir operators of the Lorentz subgroup, and the corresponding functions do not depend on x.

For decomposing the left GRR we are going to construct a complete set of commuting operators in the space of functions on the group. Together with the Casimir operators some functions of right generators[4] may be included in such a set. Therefore it is necessary to know the explicit form of right generators. As a consequence of the formulas

$$
T_R(g)f(x, \mathbf{z}) = f(xg,\ \mathbf{z}g), \quad xg \leftrightarrow X + ZAZ^\dagger, \quad \mathbf{z}g \leftrightarrow ZU, \tag{2.40}
$$

$$
T_R(g)\varphi(p, \mathbf{z}) = e^{-ia'p}\varphi(p, \mathbf{z}g), \quad a' \leftrightarrow A' = ZAZ^\dagger \tag{2.41}
$$

one can obtain

$$
\hat{p}^R_\mu = -(L^{-1}(\mathbf{z}))_\mu{}^\nu p_\nu, \quad \hat{J}^R_{\mu\nu} = \hat{S}^R_{\mu\nu}, \tag{2.42}
$$

where $L \in SO(D,1)$ (or $L \in SO(D,1)$ in the Euclidean case). The operators of right translations can also be written in the form $\hat{P}^R = -Z^{-1}\hat{P}(Z^{-1})^\dagger$; operators $\hat{S}_{\mu\nu}$ and $\hat{S}^R_{\mu\nu}$ are left and right generators of $\mathrm{Spin}(D,1)$ (or $\mathrm{Spin}(D)$) and depend on $\mathbf{z}$ only. All the right generators (2.42) commute with all the left generators (2.36) and obey the same commutation relations (2.37).

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[3] $p^2 = \eta^{\mu\nu}p_\mu p_\nu$. Since we do not use p with upper indices, this does not lead to a misunderstanding.

[4] The physical meaning of the right generators is not so transparent. However, one can remember that the right generators of $SO(3)$ in the nonrelativistic rotator theory are interpreted as operators of angular momentum in a rotating body-fixed reference frame [24–26].

In accordance with theory of harmonic analysis on Lie groups [17,16] there exists a complete set of commuting operators, which includes Casimir operators, a set of the left generators and a set of right generators (both sets contain the same number of the generators). The total number of commuting operators is equal to the number of parameters of the group. In a decomposition of the left GRR the nonequivalent representations are distinguished by eigenvalues of the Casimir operators, equivalent representations are distinguished by eigenvalues of the right generators, and the states inside the irrep are distinguished by eigenvalues of the left generators.

In particular, Casimir operators of spin Lorentz subgroup are functions of $\hat{S}^R_{\mu\nu}$ (or $\hat{S}_{\mu\nu}$) and commute with all the left generators (with left translations and rotations), but do not commute with generators of the right translations. These operators distinguish equivalent representations in the decomposition of the left GRR. Notice that some aspects of the theory of harmonic analysis on the 3+1 and 2+1 Poincaré groups were considered in [27–29] and [30] respectively.

If GRR acts in the space of all functions on the group G, then a regular representation acts in the space of functions $L^2(G,\mu)$, such that the norm

$$\int \overset{*}{f}(g)f(g)d\mu(g) \tag{2.43}$$

is finite [15,17], where $d\mu(g)$ is an invariant measure on the group. The regular representation is unitary, as it follows from (2.43) and from the invariance of the measure. However we will also use nonunitary representations (in particular, finite-dimensional representations of the Lorentz group). Therefore we consider the GRR as a more useful concept.

D. Fields on the Poincaré group

As we have shown, the relations associated with the left GRR (2.23) define the transformation law for coordinates $(x,\mathbf{z})$ on the position-spin space under the change from one Lorentz reference frame to another. The equations

$$f'(x',\mathbf{z}') = f(x,\mathbf{z}), \tag{2.44}$$
$$x' = gx = \Lambda x + a \leftrightarrow UXU^\dagger + A, \quad \mathbf{z}' = g\mathbf{z} \leftrightarrow UZ. \tag{2.45}$$

define a scalar field on this space (i.e. a scalar field on the Poincaré group). In contrast to scalar field on Minkowski space, this field is reducible with respect to both mass and spin.

Consider the transformation laws of x and $\mathbf{z}$ in various dimensions more detail.

In two-dimensional case matrices Z depend on only one parameter (angle or hyperbolic angle, see (2.16),(2.17)). The functions on the group depend on $x = (x^\mu)$ and $z = e^\alpha$ (or $x = (x^k)$ and $z = e^{i\alpha}$ in Euclidean case); it is appropriate to consider these functions as functions of real parameter α directly.

In three-dimensional case according to (2.14),(2.15)

$$D=3: \quad Z = \begin{pmatrix} z_1 & -\overset{*}{z}_{\dot 2} \\ z_2 & \overset{*}{z}_{\dot 1} \end{pmatrix}, \qquad d=2+1: \quad Z = \begin{pmatrix} z_1 & \overset{*}{z}_{\dot 2} \\ z_2 & \overset{*}{z}_{\dot 1} \end{pmatrix}, \qquad \det Z = 1. \tag{2.46}$$

Functions $f(x, \mathbf{z})$ depend on $x = (x^\mu)$ (in Euclidean case $x = (x^k)$) and $\mathbf{z} = (z, \overset{*}{z})$, where z are the elements of the first column of matrix (2.46). Let us write the relation (2.45) for $d = 2+1$ in component-wise form

$$x'^\nu \sigma_{\nu\alpha\dot\alpha} = U_\alpha{}^\beta x^\mu \sigma_{\mu\beta\dot\beta} \overset{*}{U}{}^{\dot\beta}{}_{\dot\alpha} + a^\mu \sigma_{\mu\alpha\dot\alpha}, \tag{2.47}$$

$$z'_\alpha = U_\alpha{}^\beta z_\beta, \quad \overset{*}{z}{}'_{\dot\alpha} = \overset{*}{U}{}_{\dot\alpha}{}^{\dot\beta} \overset{*}{z}_{\dot\beta}, \quad z'^\alpha = (U^{-1})^\alpha{}_\beta z^\beta, \quad \overset{*}{z}{}'^{\dot\alpha} = (\overset{*}{U}{}^{-1})^{\dot\alpha}{}_{\dot\beta} \overset{*}{z}{}^{\dot\beta}. \tag{2.48}$$

Undotted and dotted indices correspond to spinors transforming by means of matrix U and complex conjugate matrix $\overset{*}{U}$. Invariant tensor $\sigma_{\nu\alpha\dot\alpha}$ has one vector index and two spinor indices of distinct types.

For the group $M(3,1)$ matrices Z, $\det Z = 1$, has the form (2.27); the elements z^α and $\underline{z}^\alpha$ of first and second columns of matrix (2.27) are subjected to the same transformation law. The functions $f(x, \mathbf{z})$ depend on $x = (x^\mu)$ and $\mathbf{z} = (z, \overset{*}{z}, \underline{z}, \overset{*}{\underline{z}})$. The main reason to consider not real parameters (for example, real and imaginary parts of $z, \underline{z}$), but of $z, \underline{z}$ and $\overset{*}{z}, \overset{*}{\underline{z}}$, is the fact that the complex variables are subjected to simple transformation rule. Besides, the use of spaces of analytic and antianalytic functions is suitable for the problem of decomposition of GRR.

According to (2.45) and (2.22) one may write the transformation law of x^μ, z_α, $\overset{*}{z}_{\dot\alpha}$ in component-wise form

$$x'^\nu \sigma_{\nu\alpha\dot\alpha} = U_\alpha{}^\beta x^\mu \sigma_{\mu\beta\dot\beta} \overset{*}{U}{}^{\dot\beta}{}_{\dot\alpha} + a^\mu \sigma_{\mu\alpha\dot\alpha}, \qquad x'^\nu \bar\sigma_\nu{}^{\dot\alpha\alpha} = (\overset{*}{U}{}^{-1})^{\dot\alpha}{}_{\dot\beta} x^\mu \bar\sigma_\mu{}^{\dot\beta\beta} (U^{-1})_\beta{}^\alpha + a^\mu \bar\sigma_\mu{}^{\dot\alpha\alpha}, \tag{2.49}$$

$$z'_\alpha = U_\alpha{}^\beta z_\beta, \quad \overset{*}{z}{}'_{\dot\alpha} = \overset{*}{U}{}_{\dot\alpha}{}^{\dot\beta} \overset{*}{z}_{\dot\beta}, \quad z'^\alpha = (U^{-1})^\alpha{}_\beta z^\beta, \quad \overset{*}{z}{}'^{\dot\alpha} = (\overset{*}{U}{}^{-1})^{\dot\alpha}{}_{\dot\beta} \overset{*}{z}{}^{\dot\beta}. \tag{2.50}$$

It is easy to see from (2.49) that the tensors

$$\sigma_{\mu\alpha\dot\alpha} = (\sigma_\mu)_{\alpha\dot\alpha}, \quad \bar\sigma_\mu{}^{\dot\alpha\alpha} = (\bar\sigma_\mu)^{\dot\alpha\alpha} \tag{2.51}$$

are invariant. These tensors are usually used to convert vector indices into spinor ones and vice versa or to construct vector from two spinors of different types:

$$x^\mu = \frac{1}{2} \bar\sigma^{\mu\dot\alpha\alpha} x_{\dot\alpha\alpha}, \quad x_{\alpha\dot\alpha} = \sigma_{\mu\alpha\dot\alpha} x^\mu, \quad q^\mu = \frac{1}{2} \bar\sigma^{\mu\dot\alpha\alpha} z_\alpha \overset{*}{z}_{\dot\alpha}. \tag{2.52}$$

In consequence of the unimodularity of 2×2 matrices U there exist invariant antisymmetric tensors $\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}$, $\varepsilon^{\dot\alpha\dot\beta} = -\varepsilon^{\dot\beta\dot\alpha}$, $\varepsilon^{12} = \varepsilon^{\dot1\dot2} = 1$, $\varepsilon_{12} = \varepsilon_{\dot1\dot2} = -1$. Now spinor indices are lowered and raised according to the rules

$$z_\alpha = \varepsilon_{\alpha\beta} z^\beta, \quad z^\alpha = \varepsilon^{\alpha\beta} z_\beta, \tag{2.53}$$

and in particular one can get $\sigma_{\mu\alpha\dot\alpha} = \bar\sigma_{\mu\dot\alpha\alpha}$. Below we will also use the notations $\partial_\alpha = \partial/\partial z^\alpha$, $\partial^{\dot\alpha} = \partial/\partial \overset{*}{z}_{\dot\alpha}$, and correspondingly $\partial^\alpha = -\partial/\partial z_\alpha$, $\partial_{\dot\alpha} = -\partial/\partial \overset{*}{z}{}^{\dot\alpha}$.

In the framework of theory of the scalar functions on the Poincaré group *a standard spin description in terms of multicomponent functions arises under the separation of space and spin variables.*

Since $\mathbf{z}$ is invariant under translations, any function $\phi(\mathbf{z})$ carry a representation of the Lorentz group. Let a function $f(h) = f(x, \mathbf{z})$ allows the representation

$$f(x, \mathbf{z}) = \phi^n(\mathbf{z})\psi_n(x), \tag{2.54}$$

where $\phi^n(\mathbf{z})$ form a basis in the representation space of the Lorentz group. The latter means that one may decompose the functions $\phi^n(\mathbf{z}')$ of transformed argument $\mathbf{z}' = g\mathbf{z}$ in terms of the functions $\phi^n(z)$:

$$\phi^n(\mathbf{z}') = \phi^l(\mathbf{z})L_l{}^n(U). \tag{2.55}$$

An action of the Poincaré group on a line $\phi^n(\mathbf{z})\phi^n(\mathbf{z})$ is reduced to a multiplication by matrix $L(U)$, where $U \in \mathrm{Spin}(D,1)$, $\phi(\mathbf{z}') = \phi(\mathbf{z})L(U)$.

Comparing the decompositions of the function $f'(x', \mathbf{z}') = f(x, \mathbf{z})$ over the transformed basis $\phi(\mathbf{z}')$ and over the initial basis $\phi(\mathbf{z})$,

$$f'(x', \mathbf{z}') = \phi(\mathbf{z}')\psi'(x') = \phi(\mathbf{z})L(U)\psi'(x') = \phi(\mathbf{z})\psi(x),$$

where $\psi(x)$ is a column with components $\psi_n(x)$, one may obtain

$$\psi'(x') = L(U^{-1})\psi(x), \tag{2.56}$$

i.e. the transformation law of a tensor field on Minkowski space. This law correspond to the representation of the Poincaré group acting in a linear space of tensor fields as follows $T(g)\psi(x) = L(U^{-1})\psi(\Lambda^{-1}(x-a))$. According to (2.55) and (2.56), the functions $\phi(z)$ and $\psi(x)$ are transformed under contragradient representations of the Lorentz group.

For example, let us consider scalar functions on the Poincaré group $f_1(x, \mathbf{z}) = \psi_\alpha(x)z^\alpha$ and $f_2(x, \mathbf{z}) = \bar{\psi}_\alpha(x)\overset{*}{z}{}^\alpha$, which correspond to spinor representations of Lorentz group. According to (2.54) and (2.56)

$$\psi'_\alpha(x') = U_\alpha{}^\beta\psi_\beta(x), \quad \bar{\psi}'_{\dot{\alpha}}(x') = \overset{*}{U}_{\dot{\alpha}}{}^{\dot{\beta}}\bar{\psi}_{\dot{\beta}}(x). \tag{2.57}$$

The product $\psi_\alpha(x)\bar{\psi}^{*\alpha}(x)$ is Poincaré invariant.

Thus tensor fields of all spins are contained in the decomposition of the field (2.44) on the Poincaré group, and the problems of their classification and construction of explicit realizations are reduced to problem of the decomposition of the left GRR.

Notice that above we reject the phase transformations, which correspond to $U = e^{i\phi}$. This transformations of $U(1)$ group do not change space-time coordinates x, but change the phase of $\mathbf{z}$. According to (2.55) and (2.56) that leads to the transformation of phase of tensor field components $\psi_n(x)$. Taking account of this transformations means the consideration of the functions on the group $T(d)\times)\mathrm{Spin}(D,1) \times U(1)$.

E. Automorphisms of the Poincaré group and discrete transformations: P,C,T

Let us consider elements $g \leftrightarrow (A, U)$, $g_0 \leftrightarrow (X, Z)$ of the Poincaré group $M(D,1)$. It is easy to see that transformations

$$(A, U) \to (\overline{A}, (U^\dagger)^{-1}), \quad (X, Z) \to (\overline{X}, (Z^\dagger)^{-1}), \tag{2.58}$$

$$(A, U) \to (\overset{*}{A}, \overset{*}{U}), \quad (X, Z) \to (\overset{*}{X}, \overset{*}{Z}), \tag{2.59}$$

$$(A, U) \to (-A, U), \quad (X, Z) \to (-X, Z) \tag{2.60}$$

are outer involutory automorphisms of the group and generate finite group consisting of eight elements.

The automorphisms (2.58)-(2.60) define discrete transformations of space-time and spin coordinates $x, \mathbf{z}$. The substitution of transformed coordinates into the functions $f(x, \mathbf{z})$ (or into the generators (2.36)) leads to change signs of some physical variables. (Notice that the substitution both into the functions and into the generators leaves signs unaltered.)

The space reflection (or parity transformation P) is defined by the relations $x^0 \to x^0$, $x^k \to -x^k$, or $X \to \overline{X}$. If X is transformed by means of the group element (A, U), then $\overline{X}$ is transformed by means of the group element $(\overline{A}, (U^\dagger)^{-1})$, see (2.22). Therefore the space reflection represents a realization of the automorphism (2.58) of the Poincaré group

$$(X, Z) \xrightarrow{P} (\overline{X}, (Z^\dagger)^{-1}). \tag{2.61}$$

Thus, under the space reflection x and $\mathbf{z}$ have to be changed in all the constructions according to (2.61). In particular, for the momentum $P = p_\mu \sigma^\mu$ we obtain $P \to \overline{P}$, where $\overline{P} = p_\mu \bar{\sigma}^\mu$. The generators of the rotations are not changed and the generators of the boosts change their signs only.

The time reflection transformation T' is defined by the relation $x^\mu \to (-1)^{\delta_{0\mu}} x^\mu$, or $X \to -\overline{X}$, and corresponds to the composition of automorphisms (2.58) and (2.60):

$$(X, Z) \xrightarrow{T'} (-\overline{X}, (Z^\dagger)^{-1}). \tag{2.62}$$

Inversion PT', $(X, Z) \xrightarrow{PT'} (-X, Z)$, corresponds to the automorphism (2.60).

Automorphism of complex conjugation (2.59) means substitution $i \to -i$,

$$f(x, \mathbf{z}) \xrightarrow{C} \overset{*}{f}(x, \mathbf{z}). \tag{2.63}$$

One can show that in the framework of the characteristics related to the Poincaré group this transformation corresponds to the charge conjugation. Both the transformation (2.63) and charge conjugation change signs of all the generators, $\hat{p}_\mu \to -\hat{p}_\mu$, $\hat{L}_{\mu\nu} \to -\hat{L}_{\mu\nu}$, $\hat{S}_{\mu\nu} \to -\hat{S}_{\mu\nu}$. Below, considering RWE, we will see that transformation (2.63) change also the sign of current vector j^μ.

The time reversal T is defined by the relation $X \to -\overline{X}$ (the time reflection transformation T'), with the supplementary condition of energy sign conservation that means $P \to \overline{P}$. Therefore, the conditions $\hat{p}_\mu \to -(-1)^{\delta_{0\mu}} \hat{p}_\mu$, $\hat{L}_{\mu\nu} \to -(-1)^{\delta_{0\mu}+\delta_{0\nu}} \hat{L}_{\mu\nu}$, $\hat{S}_{\mu\nu} \to -(-1)^{\delta_{0\mu}+\delta_{0\nu}} \hat{S}_{\mu\nu}$ take place. The transformation CT' obeys these conditions.

However, it is known [31,32] that it is possible to give two distinct definitions of time reversal transformation obeying conditions mentioned above. Wigner time reversal T_w leaves the total charge (and correspondingly j^0) unaltered, and reverses the direction of current j^k. Schwinger time reversal T_{sch} [34] leaves the current j^k invariant and reverses the charge.

The transformation CT' changes the sign of j^0 and therefore can be identify with Schwinger time reversal, $T_{sch} = CT'$. CPT_{sch}-transformation corresponds to the inversion $(X, Z) \to (-X, Z)$. Wigner time reversal T_w and CPT_w-transformation can be defined considering both outer and inner automorphisms of the proper Poincaré group [33]. Namely, $CPT_w = I_x I_z$, where I_z is defined as

$$(X, Z) \xrightarrow{I_z} (X, Z(-i\sigma_2)) \tag{2.64}$$

and is a composition of the inner automorphism $(X, Z) \to (\overline{X}^T, (Z^T)^{-1})$ and of the rotation by the angle π. Wigner time reversal is the composition of above considered transformations, $T_w = I_z CT = I_z T_{sch}$.

The improper Poincaré group is defined as a group, which includes continuous transformations of the proper Poincaré group $g \in M(D, 1)$ and the space reflection P.

In the Euclidean case the space reflection is reduced to the substitution $(X, Z) \xrightarrow{P} (-X, Z)$. The charge conjugation inverts the momentum and spin orientation.

F. Equivalent representations

In the decomposition of scalar field (2.44) on the Poincaré group (or, that is the same, of the left GRR) there are equivalent representations distinguished by the right generators.

Remember that representations $T_1(g)$ and $T_2(g)$ acting in linear spaces L_1 and L_2 respectively are equivalent if there exists an invertible linear operator $A : L_1 \to L_2$ such that

$$AT_1(g) = T_2(g)A. \tag{2.65}$$

In particular, the left and the right GRR of a Lee group G are equivalent. The operator $(Af)(g) = f(g^{-1})$ realizes the equivalence [17,14].

Let us consider functions $f(x, \mathbf{z})$ belonging to two equivalent representations in the decomposition of the left GRR of the group $M(D, 1)$ (or $M(D)$). If the representations $T_1(g)$ and $T_2(g)$ acting in the different subspaces L_1 and L_2 of the space of functions on the group are equivalent, then

$$AT_1(g)f_1(x, \mathbf{z}) = T_2(g)Af_1(x, \mathbf{z}), \quad f_2(x, \mathbf{z}) = Af_1(x, \mathbf{z}),$$

where $f_1(x, \mathbf{z}) \in L_1$ and $f_2(x, \mathbf{z}) \in L_2$. In particular, if operator $A : L_1 \to L_2$ is a function of the right translations generators $\hat{p}^R_\mu$, then one can't map the function $f_1(x, \mathbf{z})$ to the function $f_2(x, \mathbf{z})$ by the group transformation, which leaves the interval square invariant. Therefore the physical equivalence of the states, that correspond to equivalent irreps in the decomposition of the scalar field $f(x, \mathbf{z})$, is not evident at least.

Below we will consider a number of examples in various dimensions. In particular, in the framework of the representation theory of three-dimensional Euclidean group $M(3)$ irreps characterized by different spins (but with the same spin projection on the direction of propagation) are equivalent. There are no contradictions in the fact that in this case different particles are described by equivalent irreps since it is not possible to map corresponding wave functions one into another by the rotations or translations of the frame of references.

In some cases more general consideration may be based on the representation theory of an extended group. In the framework of the latter there are two possibilities: either irreps labelled by different eigenvalues of right generators of initial group are nonequivalent or some equivalent irreps of initial group are combined into one irrep. For example, in nonrelativistic theory spin becomes the characteristic of nonequivalent irreps after the extantion of $M(3)$ up

to Galilei group. In 3+1 dimensions for $m > 0$ the proper Poincaré group representations characterized by different chiralities are equivalent. If we going from the Lorentz group to the group $SO(3,2)$, then all characterized by spin s states with different chiralities λ, $\lambda = -s, -s+1, \ldots, s$ are combined into one irrep.

The space of functions $f(x, \mathbf{z})$ contains functions transforming under equivalent representations of the proper Poincaré group and is sufficiently wide to define discrete transformations, including space reflection, time reflection, and charge conjugation. These discrete transformations associated with automorphisms of the group also combine equivalent irreps of proper Poincaré group into one representation of the extended group. For example, in 3+1 dimensions space reflection combines two equivalent irreps of the proper group labelled by λ and $-\lambda$ into one irrep of the improper group.

Besides, as we will see below, the different types of RWE (finite-component and infinite-component equations) are also associated with equivalent representations in the decomposition of the left GRR.

Thus initially it is appropriate to consider all representations in the decomposition of the scalar field on the Poincaré group, including equivalent ones. In this sense we note the close analogy with the theory of nonrelativistic three-dimensional rotator [24–26]. In the latter theory one considers functions on the rotation group $SU(2)$ and two sets of operators: angular momentum operators in an inertial laboratory (space-fixed) frame (left generators $\hat{J}^L_i$) and angular momentum operators in a rotating (body-fixed) frame (right generators $\hat{J}^R_i$). The classification of the rotator states is based on the use of the complete set of commuting operators which, apart from $\hat{\mathbf{J}}^2$ and $\hat{J}^L_3$, includes also $\hat{J}^R_3$. Operator $\hat{J}^R_3$ distinguishes equivalent representations in the decomposition of the left GGR of the rotation group and corresponds to the quantum number which does not depend on the choice of the laboratory frame. This quantum number plays a significant role in the theory of molecular spectra. In 3+1-dimensional case there exist two analogs of $\hat{J}^R_3$, namely $\hat{B}^R_3 = \hat{S}^R_{03}$ and $\hat{S}^R_3 = \hat{S}^R_{12}$, which act in the space of functions on the Poincaré group. As we will see below, the first may be interpreted as a chirality operator, and the second allows to distinguish particles and antiparticles.

G. Quasiregular representations and spin description

The consideration of GRR of the Poincaré group ensures the possibility of consistent description of particles with arbitrary spin by means of scalar functions on $\mathbb{R}^d \times \mathrm{Spin}(D,1)$. At the same time, for description of spinning particles it is possible to use the spaces $\mathbb{R}^d \times M$, where M is some homogeneous space of the Lorentz group (one or two-sheeted hyperboloid, cone, complex disk, projective space and so on); see, for example, [7–9,35–41,13] and [42–45] for 3+1 and 2+1-dimensional cases respectively. In some papers fields on homogeneous spaces are considered; in other papers such spaces are treated as phase spaces of some classic mechanics, and the latter are treated as models of spinning relativistic particles.

These spaces appear in the framework of the next group-theoretical scheme. Let us consider the left quasiregular representation of the Poincaré group

$$T(g)f(g_0H) = f(g^{-1}g_0H), \quad H \subset \mathrm{Spin}(D,1). \tag{2.66}$$

H is a subgroup of $\mathrm{Spin}(D,1)$, and since x is invariant under right rotations (see (2.40))

$$g_0 \leftrightarrow (X,Z), \quad g_0 H \leftrightarrow (X, ZH).$$

Therefore the relation (2.66) defines the representation of the Poincaré group in the space of functions $f(x, zH)$ on

$$\mathbb{R}^d \times (\mathrm{Spin}(D,1)/H). \tag{2.67}$$

In the decomposition of the representation in the space of functions on $\mathrm{Spin}(D,1)/H$ (or $\mathbb{R}^d \times (\mathrm{Spin}(D,1)/H)$) there is, generally speaking, only part of irreps of the Lorentz (or Poincaré) group. In particular, the case $H \sim \mathrm{Spin}(D,1)$ corresponds to scalar field on Minkowski space. The classification and description of homogeneous spaces of 3+1 Poincaré and Lorentz groups one can find in [5,7,46].

Thus the consideration of quasiregular representations allows one to construct a number of spin models classified by subgroups $H \subset \mathrm{Spin}(D,1)$. But the existence of nontrivial subgroup H leads to rejection of the part of equivalent (with different characteristics with respect to the Lorentz subgroup) or, possibly, nonequivalent irreps of the Poincaré group.

H. Relativistic wave equations

The problem of RWE construction for particles with arbitrary spin in various dimensions is far from its completion and continues to attract significant attention. To describe massive particles of spin s in four dimensions one usually employs the equations connected with the representations $(\frac{s}{2}\,\frac{s}{2})$ and $(\frac{2s\pm1}{4}\,\frac{2s\mp1}{4})$ of the Lorentz group (see, for example, [21,47]). These equations admit Lagrangian formulations [48–50], but for $s>1$ minimal electromagnetic coupling leads to a noncasual propagation [51,52]. On the other hand, all known equations with casual solutions either have a redundant number of independent components (as equations [53,54] for representations $(s\,0)$ and $(0\,s)$ have) or describe many masses and spins simultaneously, as Bhabha equations [55–57] do. Besides the problem of interaction of higher spin fields, one may mention attempts to construct RWE with a completely positive energy spectrum [58–62] and RWE for fractional spin fields [42–44,30].

With respect to mathematical methods used, it is possible divide all approaches to RWE construction in three groups.

The first approach, which follows Refs. [63,48,64,2], deals with equations for symmetric spin tensors. It allows one to describe fields with fixed mass and spin and also to construct RWE which admit Lagrangian formulation; however, as was mentioned above, for $s>1$ we face the problem of noncasual propagation.

The second approach, which follows Refs. [65,55,56,66,67,59], is devoted to studying RWE of the form $(\alpha^\mu \hat{p}_\mu - \varkappa)\psi(x) = 0$, and is based on the use of algebraic properties of α-matrices. These equations admit Lagrangian formulation, however, for $s>1$ they describe a nonphysical spectrum of particles: a decreasing mass with increasing spin.

The third approach is connected to the use of some supplementary variables to describe spin degrees of freedom and initially was suggested for RWE with a mass spectrum (see [1,68]). It was used for constructing positive energy wave equations [60–62], equations describing gauge fields [69,70], and for anyon equations [42–44,30].

From the point of view of the approach which we developed above, the problem of constructing RWE looks like a selection of invariant subspaces in the space of functions on the group.

The classification of the scalar functions can be based on the use of the operators $\hat{C}_k$ commuting with $T_L(g)$ (and correspondingly with all the left generators). For these operators, as a consequence of a relation $\hat{C}f(x,\mathbf{z}) = cf(x,\mathbf{z})$, one can obtain that $\hat{C}f'(x,\mathbf{z}) = cf'(x,\mathbf{z})$, where $f'(x,\mathbf{z}) = T_L(g)f(x,\mathbf{z})$. Therefore, different eigenvalues c correspond to subspaces, which are invariant with respect to action of $T_L(g)$. The invariant subspaces correspond to subrepresentations of the left GRR.

In addition to the Casimir operators, for the classification one may use the right generators since all the right generators commute with all the left generators. The right generators, as was mentioned, distinguish equivalent representations in the decomposition of the left GRR.

There is some freedom to choose the commuting operators which are functions of the right generators of the Poincaré group. We will use only functions of the generators of the right rotations (2.42), in particular, for the coordination with standard formulation of the theory.

Following the general scheme of harmonic analysis, for $M(D,1)$ one may consider the system consisting of d equations

$$\hat{C}_k f(x,\mathbf{z}) = c_k f(x,\mathbf{z}), \tag{2.68}$$

where $\hat{C}_k$ are the Casimir operators of the Poincaré group and of the spin Lorentz subgroup. These operators constitute a subset of the complete set of commuting operators on the Poincare group. Just the system we will use for $d = 2+1$ below.

On the other hand, there exist some additional requirements associated with the physical interpretation. In the first place, in massive case the system must be invariant under space reflection in order to describe states with definite parity. Secondly, it is often supposed that the system contains an equation of the first order in $\partial/\partial t$ (approach based on the first order equations advocated mainly in [56,71,72]).[5]

The Casimir operators of the Poincaré group are the functions of the generators $\hat{p}_\mu$ and $\hat{J}_{\mu\nu}$. In odd dimensions there exists linear in $\hat{p}_\mu$ Casimir operator, since the invariant tensor $\varepsilon^{\mu\ldots\nu}$ has also odd number of indices. As we will see below, in 2+1 dimensions the system (2.68) is invariant under space reflections.

In even dimensions the invariant tensor $\varepsilon^{\mu\ldots\nu}$ also has even number of indices, and therefore linear in $\hat{p}_\mu$ Casimir operator does not exist. Besides, in even dimensions under space reflection irrep of proper Poincaré group is mapped onto equivalent irrep labelled by another eigenvalues of the Casimir operators of spin Lorentz subgroup. The linear combinations of basis elements of these two irreps form the bases of two labelled by intrinsic parity $\eta = \pm 1$ irreps of improper Poincaré group including space reflection.

[5] As a consequence of relativistic invariance, a linear in $\partial/\partial t$ equation can be either first order or infinite order in space derivatives (square-root Klein-Gordon equation [73–76]). The latter type of equations are naturally obtained in the theory of Markov processes for probability amplitudes [77].

Nevertheless, in even dimensions there exists operator $\hat{C}' = \hat{p}_\mu \hat{\Gamma}^\mu$, where $\hat{\Gamma}^\mu = \hat{\Gamma}^\mu(\mathbf{z}, \partial/\partial\mathbf{z})$, commuting with all left generators and connecting the states which are interchanged under space reflections. In contrast to the Casimir operators this operator is not a function of generators of Poincaré group and does not commute with some right generators. A first order equation

$$\hat{p}_\mu \hat{\Gamma}^\mu f(x, \mathbf{z}) = \varkappa f(x, \mathbf{z}) \tag{2.69}$$

interlocks, at least, two irreps of the group $M(D, 1)$ characterized by different eigenvalues of the Casimir operator of spin Lorentz subgroup. Equations (2.68) and (2.69) have the same form; namely, invariant operator acts on the scalar function $f(x, \mathbf{z})$ on the group $M(D, 1)$. The addition of the operators $\hat{\Gamma}^\mu$ means in fact the extension of the Lorentz group up to more wide group (in particular, in four dimensions to the 3+2 de Sitter group $SO(3, 2)$). Equation (2.69) replaces the equations of the system (2.68), which are not invariant under space reflection.

In the approach under consideration equations have the same form for all spins. The separation of the components with fixed spin and mass is realized by fixing eigenvalues of the Casimir operators of the Poincaré group (or the operator $\hat{p}_\mu \hat{\Gamma}^\mu$). Fixing the representation of the Lorentz group, under which $\phi(\mathbf{z})$ transforms in the decomposition

$$f(x, \mathbf{z}) = \phi^n(\mathbf{z}) \psi_n(x),$$

one can obtain RWE in standard multicomponent form. This fixation is realized by the Casimir operator of spin Lorentz subgroup.

There are two types of equations to describe one and the same spin, one on functions $f(x, \mathbf{z})$, where $\phi^n(\mathbf{z})$ transforms under finite-dimensional nonunitary irrep of the Lorentz group, and another on functions $f(x, \mathbf{z})$, where $\phi^n(\mathbf{z})$ transforms under infinite-dimensional unitary irrep of the Lorentz group. In matrix representation these equations are written in the form of finite-component or infinite-component equations respectively. The latter type of equations (for example, Majorana equations [58,59,78,60]) is interesting because it gives the possibility to combine the relativistic invariance with probability interpretation. Desirability of this combination was emphasized in [79].

Let us briefly consider the possibility of existence of particles with fractional spin. The restrictions on the spin value arise in the representation theory of $M(D)$ and $M(D, 1)$ if one restricts the consideration by (1) unitary, (2) finite-dimensional (with respect to the number of spin components) or (3) single-valued representations. (The latter means that the representation acts in the space of single-valued functions.) The restriction by single-valued functions (often supposed in mathematical papers related to a classification of representations) is omitted in some physical problems that allows to consider particles with fractional spin (anyons). Thus, we will also consider multi-valued representations of $M(D)$ and $M(D, 1)$ in the space of the functions $f(x, \mathbf{z})$ on the group. These representations correspond to single-valued representations of the universal covering group.

III. TWO-DIMENSIONAL CASE

A. Field on the group $M(2)$

In two-dimensional case the general formulas become simpler. Matrices U (2.17) of $SO(2)$ subgroup depend on only one parameter, namely an angle ϕ, $0 \le \phi \le 4\pi$. Using the correspondence $g_0 \leftrightarrow (X, Z(\theta/2))$, $g \leftrightarrow (A, U(\phi/2))$ one may write the action of GRR:

$$T_L(g)f(x,\theta/2) = f(x',\theta/2-\phi/2), \tag{3.1}$$
$$x'_1 = (x_1-a_1)\cos\phi + (x_2-a_2)\sin\phi, \quad x'_2 = (x_2-a_2)\cos\phi - (x_1-a_1)\sin\phi,$$
$$T_R(g)f(x,\theta/2) = f(x'',\theta/2+\phi/2), \tag{3.2}$$
$$x''_1 = x_1 + a_1\cos\theta - a_2\sin\theta, \quad x''_2 = x_2 + a_2\cos\theta + a_1\sin\theta.$$

Left and right generators, which correspond to parameters θ and ϕ, are given by

$$\hat{p}_k = -i\partial_k, \quad \hat{J} = \hat{L} + \hat{S}, \tag{3.3}$$
$$\hat{p}^R_k = i\Lambda_k{}^i\partial_i, \quad \hat{J}^R = -\hat{S}, \tag{3.4}$$

where

$$\hat{L} = i(x_1\partial_2 - x_2\partial_1) = -i\frac{\partial}{\partial\varphi}, \quad \hat{S} = -i\frac{\partial}{\partial\theta}, \quad \Lambda = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

The functions on the group are ones on $\mathbb{R}^2 \times S^1$, and invariant measure on the group is

$$d\mu(x,\theta) = (4\pi)^{-1}dx_1dx_2d\theta, \quad -\infty < x < +\infty, \quad 0 \le \theta < 4\pi.$$

We will consider two complete sets of commuting operators: $\hat{p}_1$, $\hat{p}_2$, $\hat{S}$ and $\hat{p}^2$, $\hat{J}$, $\hat{S}$. The eigenfunctions of these operators are

$$\langle x_1x_2\theta|p_1\,p_2\,s\rangle = (2\pi)^{-1}\exp(ip_1x_1 + ip_2x_2 + is\theta), \tag{3.5}$$
$$\langle r\varphi\theta|p\,j\,s\rangle = (2\pi)^{-1/2}i^lJ_l(pr)\exp(il\varphi)\exp(is\theta), \tag{3.6}$$

where $l = j - s$ is orbital momentum, $J_l(pr)$ is the Bessel function. Irreps are labelled by eigenvalues p^2 of the Casimir operator $\hat{p}^2$. For $p \ne 0$ the representation is irreducible, for $p = 0$ it decomposes into one-dimensional irreps of spin subgroup $U(1)$, which are labelled by eigenvalues s of the spin projection operator (or, simply speaking, spin operator) $\hat{S}$.

At $p \ne 0$ the representations characterized by the spin s and $s' = s+n$, where n is integer number, are equivalent. Really, operator $\hat{S}$ commutes with all left generators, but does not commute with the generators of right translations, which mix spin and space coordinates. Operators $\hat{p}^R_+ = p^R_1 - ip^R_2$ and $\hat{p}^R_- = p^R_1 + ip^R_2$ are raising and lowering operators with respect to spin s

$$\hat{p}^R_\pm|p_1\,p_2\,s\rangle = (ip_1 \pm p_2)|p_1\,p_2\,s\pm 1\rangle. \tag{3.7}$$

Right translations do not conserve both interval (distance) and spin s.

The functions (3.6) satisfy the relations of orthogonality and completeness

$$\int \langle p\,j\,s|r\varphi\theta\rangle\langle r\varphi\theta|p\,j\,s\rangle rdrd\varphi d\theta = \frac{\delta(p-p')}{p}\delta_{jj'}\delta_{ss'}, \tag{3.8}$$
$$\int\sum_{l,s}\langle r\varphi\theta|p\,j\,s\rangle\langle p\,j\,s|r\varphi\theta\rangle dp = \frac{\delta(r-r')}{r}\delta(\varphi-\varphi')\delta(\theta-\theta'). \tag{3.9}$$

It means that we have obtained the decomposition of left regular representation. Spin operator $\hat{S}$ distinguishes equivalent irreps (except the case $p = 0$, when irreps are labelled by its eigenvalues). The decomposition of the functions of θ on the eigenfunctions of $\hat{S}$ corresponds to the Fourier series expansion of functions on a circle.

Thus the representations of $M(2)$ are single-valued for integer and half-integer s. The fractional values of s correspond to multi-valued representations. Irreps are equivalent if are labelled by the same $p \neq 0$ and the difference $s - s' = n$ is an integer number. For fixed $p \neq 0$ there are only two nonequivalent single-valued representations, which correspond to integer and half-integer spin. Nonequivalent multi-valued representations for fixed $p \neq 0$ are labelled by $\tilde{s} \in [0, 1)$, $\tilde{s} = s - [s]$.

B. Field on the group $M(1,1)$

Matrices U (2.16) of $SO(1,1)$ subgroup, which is isomorphic to an additive group of real numbers, depend on a hyperbolic angle ϕ. Using the correspondence $g_0 \leftrightarrow (X, Z(\theta/2))$, $g \leftrightarrow (A, U(\phi/2))$, one may write the action of GRR:

$$T_L(g)f(x, \theta/2) = f(x', \theta/2 - \phi/2), \tag{3.10}$$
$$x'^0 = (x^0 - a^0)\cosh\phi + (x^1 - a^1)\sinh\phi, \quad x'^1 = (x^1 - a^1)\cosh\phi + (x^0 - a^0)\sinh\phi,$$
$$T_R(g)f(x, \theta/2) = f(x'', \theta/2 + \phi/2), \tag{3.11}$$
$$x''^0 = x^0 + a^0\cosh\theta - a^1\sinh\theta, \quad x''^1 = x^1 + a^1\cosh\theta - a^0\sinh\theta.$$

The functions on the group are ones on $\mathbb{R}^2 \times \mathbb{R}$, and invariant measure on the group can be written as

$$d\mu(x, \theta) = dx^0 dx^1 d\theta, \quad -\infty < x, \theta < +\infty.$$

As above, we will consider two complete sets of commuting operators, $\hat{p}_1$, $\hat{p}_2$, $\hat{S}$ and $\hat{p}^2$, $\hat{J}$, $\hat{S}$, where $\hat{J} = \hat{L} + \hat{S}$, $\hat{L} = i(x^0\partial^0 + x^1\partial^1)$, $\hat{S} = -i\partial/\partial\theta$. The eigenfunctions of the first set are

$$\langle x^0 x^1 \theta | p_1\, p_2\, \lambda\rangle = (2\pi)^{-3/2}\exp(ip_\mu x^\mu + i\lambda\theta), \tag{3.12}$$

where λ is an eigenvalue of the spin projection (chirality) operator $\hat{S}$. The form of eigenfunctions of the second set depends on the type of irrep. There are four types of unitary irreps labelled by eigenvalue m^2 of operator $\hat{p}^2$ [80].

1. $m^2 > 0$. Representations correspond to the particles of nonzero mass, the eigenfunctions of operators $\hat{p}^2$, $\hat{J}$, $\hat{S}$ are

$$\langle r\varphi\theta | m\, j\, \lambda\rangle = (4\pi)^{-1} i \exp(\pi l/2) H^{(2)}_{il}(\pm mr)\exp(il\varphi)\exp(i\lambda\theta), \tag{3.13}$$

where $H^{(2)}_{il}(mr)$ is Hankel function, $r^2 = (x^0)^2 - (x^1)^2$, and $\pm$ corresponds to the sign of energy p_0.

2. $m^2 < 0$. Representations correspond to tachyons, which in $d = 1 + 1$ are more similar to usual particles because of symmetry between space and time variables. The form of $\langle r\varphi\theta | m\, j\, \lambda\rangle$ coincides with (3.13), but m is imaginary.

3. $m = 0$, $p_1 = \pm p_0$. Representations correspond to the massless particles. According to (2.39), for the action of finite transformations $T_0(g)$ on the functions $f(p, \pm p, \theta/2)$ one may obtain

$$T_0(g)f(p, \pm p, \theta/2) = e^{iap'} f(p',\, \pm p',\, \theta/2 - \phi/2), \quad p' = e^{\mp\phi}p.$$

Therefore the representation $T_0(g)$ is reducible and splits into four irreps differed by the signs of p_0 and $p_1 = \pm p_0$, and reducible representation, which corresponds to $m = p_0 = 0$.

4. $m = p_0 = 0$. This representation decomposes into sum of one-dimensional irreps of the group $SO(1,1)$, which are labelled by eigenvalues of $\hat{S}$.

There are no integer value restrictions for the spectrum of $\hat{S}$, and chirality can be fractional, $-\infty < \lambda < +\infty$. The decomposition of the functions $f(x, \theta)$ in therms of the eigenfunctions of $\hat{S}$ corresponds to the Fourier integral expansion of functions on a line. The equivalence of the representations characterized by different λ is related to the fact that, like in Euclidean case, operator $\hat{S}$ does not commute with right translations.

One can convert vector indices into spinor indises and vice versa with the help of the formula (2.10). In the case under consideration matrices U are real and symmeric, $X' = UXU$, or in component-wise form $x'^{\nu}\sigma_{\nu\alpha\alpha'} = U_{\alpha}{}^{\beta}\sigma_{\mu\beta\beta'}x^{\mu}U^{\beta'}{}_{\alpha'}$, and there exists one type of spinor indeces only. Denoting elements of the first column of matrix Z transforming under spinor representation of $SO(1,1)$ by z_α, $z_1 = \cosh(\theta/2)$, $z_2 = \sinh(\theta/2)$, we obtain for components of vector and antisymmetric tensor

$$q^{\mu} = \sigma^{\mu\alpha\beta}z_\alpha z_\beta, \quad q^0 = \cosh\theta, \quad q^1 = \sinh\theta, \quad q^{01} = \sigma^{01\alpha\beta}z_\alpha z_\beta = i. \tag{3.14}$$

There exist two invariant tensors $\eta^{\mu\nu}$ and $\varepsilon^{\mu\nu}$, which can be used for raising of indices. This is related to the fact that vectors $(x^0\, x^1)$ and $(x^1\, x^0)$ have the same transformation rule, and one can construct invariant from two vectors by two different ways: $\eta^{\mu\nu}q_\mu q'_\nu = \cosh(\theta - \theta')$, $\varepsilon^{\mu\nu}q_\mu q'_\nu = \sinh(\theta - \theta')$.

C. Relativistic wave equations in 1+1 dimensions

An irrep of the group $M(1,1)$ can be extract from GRR by fixing the sign of p_0 and eigenvalues of operators $\hat{p}^2$, $\hat{S}$,

$$\hat{p}^2 f(x,\theta) = m^2 f(x,\theta), \tag{3.15}$$

$$\hat{S}f(x,\theta) = \lambda f(x,\theta), \tag{3.16}$$

where chirality λ distinguishes equivalent irreps labelled by identical eigenvalues m^2 of the Casimir operator $\hat{p}^2$. Solutions of this system have the form $f(x,\theta) = \psi(x)e^{i\lambda\theta}$, where $\hat{p}^2\psi(x) = m^2\psi(x)$.

According to (2.61), space reflection converts $e^{i\lambda\theta}$ to $e^{-i\lambda\theta}$. Irreps of the improper Poincaré group are labelled by mass m, $\operatorname{sign} p_0$, intrinsic parity $\eta = \pm 1$, and spin $s = |\lambda|$ (as above, s distinguishes equivalent irreps). In the rest frame it is easy to write down functions with mentioned characteristics:

$$e^{\pm imx^0}(e^{i\lambda\theta} \pm e^{-i\lambda\theta}). \tag{3.17}$$

States with arbitrary momentum can be obtained from (3.17) by hyperbolic rotations and form the basis of unitary irrep of improper group. On the other hand, the problem arise to construct equations that unlike the system (3.15)-(3.16) are invariant under improper Poincaré group and have solutions with definite parity. These equations should combine states with chiralities $\pm\lambda$.

The general form of the linear in $\hat{p}^\mu$ equations is

$$\hat{p}_\mu \hat{\Gamma}^\mu f(x,\theta) = \varkappa f(x,\theta), \tag{3.18}$$

where $\hat{\Gamma}^\mu = \hat{\Gamma}^\mu(\theta, \partial/\partial\theta)$. For invariance of (3.18) under space reflection P and hyperbolic rotations the operator $\hat{p}_\mu \hat{\Gamma}^\mu$ must commute with P and $\hat{J} = \hat{L} + \hat{S}$, whence

$$\hat{\Gamma}^\mu \xrightarrow{P} (-1)^{\delta_{1\mu}} \hat{\Gamma}^\mu, \qquad [\hat{\Gamma}^0, \hat{S}] = i\hat{\Gamma}^1, \qquad [\hat{\Gamma}^1, \hat{S}] = i\hat{\Gamma}^0. \tag{3.19}$$

The operators

$$\hat{\Gamma}^0 = s\cosh\theta - \sinh\theta \frac{\partial}{\partial\theta}, \quad \hat{\Gamma}^1 = s\sinh\theta - \cosh\theta \frac{\partial}{\partial\theta}, \quad [\hat{\Gamma}^0, \hat{\Gamma}^1] = -i\hat{S} \tag{3.20}$$

obey these relations. One may construct the operators, which raise and lower chirality λ by 1,

$$\hat{\Gamma}_+ = \hat{\Gamma}^0 - \hat{\Gamma}^1 = e^{-\theta}(s + \partial/\partial\theta), \quad \hat{\Gamma}_- = \hat{\Gamma}^0 + \hat{\Gamma}^1 = e^{\theta}(s - \partial/\partial\theta). \tag{3.21}$$

Operators $\hat{\Gamma}^0$, $\hat{\Gamma}^1$, and $\hat{\Gamma}^2 = -i\hat{S} = -\partial/\partial\theta$ obey the commutation relations of the generators of the $SO(2,1) \sim SU(1,1)$ group:

$$[\hat{\Gamma}^a, \hat{\Gamma}^b] = \epsilon^{abc}\hat{\Gamma}_c, \quad \hat{\Gamma}_a = \eta_{ab}\hat{\Gamma}^b, \quad \eta_{00} = \eta_{22} = -\eta_{11} = 1, \quad \hat{\Gamma}_a\hat{\Gamma}^a = s(s+1).$$

Thus, if the symmetry with respect to the space reflection takes place, the condition of mass irreducibility (3.15) can be supplemented by equation (3.18) instead of (3.16). This means the passage to the new set of commuting operators, namely from $\hat{p}_\mu$, $\hat{S}$ to $\hat{p}_\mu$, $\hat{p}_\mu\hat{\Gamma}^\mu$. Let us consider the system

$$\hat{p}^2 f(x,\theta) = m^2 f(x,\theta), \tag{3.22}$$

$$\hat{p}_\mu \hat{\Gamma}^\mu f(x,\theta) = ms f(x,\theta). \tag{3.23}$$

The operator $\hat{S}$ does not commute with $\hat{p}_\mu\hat{\Gamma}^\mu$, and the particle with nonzero mass described by equation (3.23) can't be characterized by certain chirality. In the rest frame $p_0 = \pm m$, and the functions $f(x,\theta) = e^{\pm imx^0}\phi(\theta)$ should be eigenfunctions of operator $\hat{\Gamma}^0$ with eigenvalues $\pm s$. The equation

$$\hat{\Gamma}^0\phi(\theta) = (s\cosh\theta - (\sinh\theta)\partial/\partial\theta)\phi(\theta) = \varkappa\phi(\theta)$$

for $\varkappa = \pm s$ has solutions $[\cosh(\theta/2)]^{2s}$ and $[\sinh(\theta/2)]^{2s}$ respectively. Below we will consider two cases.

1. The solutions of the system (3.22)-(3.23) are sought in the space of polynomials of $e^{-\theta/2}$ and $e^{\theta/2}$ that correspond to finite-dimensional nonunitary representations of $SU(1,1)$.

Corresponding representations of $SO(1,1)$ subgroup are also nonunitary. For these representations generator $\hat{S}$ is anti-Hermitian, and it is convenient to redefine chirality operator as $i\hat{S}$. In the rest frame a general solution of the system (3.22)-(3.23) is

$$f(x,\theta) = C_1 e^{imx^0}[\cosh(\theta/2)]^{2s} + C_2 e^{-imx^0}[\sinh(\theta/2)]^{2s}, \tag{3.24}$$

where $2s$ is integer positive number. Therefore for an unique spin s there are only two independent components (with positive and negative frequency). The space inversion takes θ to $-\theta$, and in the rest frame solutions with different sign of p_0 and half-integer s are characterized by opposite parity η. For integer s all solutions are characterized by $\eta = 1$. Plane wave solutions, which correspond to moving particle, can be obtained from (3.24) by a hyperbolic rotation by the angle 2ϕ:

$$f_{m,s}(x,\theta) = C_1 e^{ik_0x^0+ik_1x^1}\left(\cosh[(\theta+\phi)/2]\right)^{2s} + C_2 e^{-ik_0x^0-ik_1x^1}\left(\sinh[(\theta+\phi)/2]\right)^{2s},$$

where $k_0 = m\cosh 2\phi$, $k_1 = m\sinh 2\phi$.

In the ultrarelativistic limit $\phi \to \pm\infty$ we have two states with chirality $\lambda = \pm s$ respectively. Thus, if in the rest frame one may distinguish two components with positive and negative frequency, then in massless limit one may distinguish two components with positive and negative chirality.

Matrix form of the system (3.22)-(3.23) can be obtained by the decomposition of $f(x,\theta)$ over the basis $e^{\lambda\theta/2}$, $\lambda = -s, -s+1, \ldots, s$. There are $2s+1$ components $\psi(x)$ in this form, but only two of them are independent. Notice that representations of $SO(1,1)$ of the form $e^{\lambda\theta}$, are nonunitary for real λ and integral over θ is divergent. One can redifine the norm of a state with the help of scalar product in the space of multicomponent functions $\psi(x)$, but this product is not positive definite.

For $s = 1/2$, substituting the function $f(x,\theta) = \psi_1(x)e^{\theta/2} + \psi_2(x)e^{-\theta/2}$ into equation (3.23), we obtain two-dimensional Dirac equation [81]

$$\hat{p}_\mu\gamma^\mu\Psi(x) = m\Psi(x), \qquad \gamma^0 = \sigma_1, \quad \gamma^1 = -i\sigma_2, \quad 2\hat{S} = \gamma^3 = \sigma_3. \tag{3.25}$$

where $\Psi(x) = (\psi_1(x)\ \psi_2(x))^T$. Matrix $\gamma^3 = \gamma^0\gamma^1$ corresponds to chirality operator and satisfies the condition $[\gamma^3, \gamma^\mu]_+ = 0$. On the other hand, this matrix corresponds to hyperbolic rotation, and similarly to 3+1 case one can write $\gamma^\mu\gamma^\nu = \eta^{\mu\nu} - i\sigma^{\mu\nu}$, where $\sigma^{01} = i\gamma^3$. Invariant scalar product has the form $|\psi_1(x)|^2 - |\psi_2(x)|^2$.

For $s = 1$, substituting the function $f(x,\theta) = \psi_{11}(x)e^{\theta} + \psi_{12}(x) + \psi_{22}(x)e^{-\theta}$ into equation (3.23), we obtain

$$\begin{aligned}&(\hat{p}_\mu\Gamma^\mu - m)\Psi(x) = 0,\\ &\Gamma^0 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}, \quad \Gamma^1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&-1&0\\1&0&-1\\0&1&0\end{pmatrix}, \quad \hat{S} = \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix},\end{aligned} \tag{3.26}$$

where $\Psi(x) = (\psi_{11}(x)\ \psi_{12}(x)/\sqrt{2}\ \psi_{22}(x))^T$. Using (3.14) to convert spinor indices to vector ones: $\mathcal{F}_0 = \psi_{22} - \psi_{11}$, $\mathcal{F}_1 = \psi_{22} + \psi_{11}$, $F_{01} = -F_{10} = -i\psi_{12}$, we obtain $p_0\mathcal{F}_1 - p_1\mathcal{F}_0 = -imF_{01}$, $ip_0F_{10} = m\mathcal{F}_1$, $ip_1F_{10} = m\mathcal{F}_0$. Thus one can rewrite 1+1 "Duffin–Kemmer" equation (3.26) in the form, which is similar to Proca equations for in 3+1 dimensions (see (5.85),(5.89)),

$$\partial_\mu \mathcal{F}_\nu - \partial_\nu \mathcal{F}_\mu = m F_{\mu\nu}, \qquad \partial^\nu F_{\mu\nu} = m \mathcal{F}_\mu. \tag{3.27}$$

As a consequence of (3.27) we obtain $\partial_\mu \mathcal{F}^\mu = 0$, $(\hat{p}^2 - m^2)\mathcal{F}^\mu = 0$. But 1+1-dimensional case is distinctly different from 3+1-dimensional one because the component $F_{01} = -F_{10}$ is characterized by zero chirality and thus the roles of $F_{\mu\nu}$ and $\mathcal{F}_\nu$ are interchanged.

In the massless case the system (3.27) splits into two independent equations for the components $\mathcal{F}_\mu$ and $F_{\mu\nu}$ respectively,

$$\partial_\mu \mathcal{F}_\nu - \partial_\nu \mathcal{F}_\mu = 0, \tag{3.28}$$

$$\partial^\mu F_{\mu\nu} = 0. \tag{3.29}$$

First equation has propagating solutions

$$\mathcal{F}_0 = C_1 e^{ip(x^0+x^1)} + C_2 e^{ip(x^0-x^1)}, \qquad \mathcal{F}_1 = C_1 e^{ip(x^0+x^1)} - C_2 e^{ip(x^0-x^1)}$$

obeying transversality condition $\partial_\mu \mathcal{F}^\mu = 0$. Second equation (free two-dimensional Maxwell equation [81]) corresponds to the components with zero chirality and has trivial solution $F_{\mu\nu} = const$ only. Notice that for real field $f^*(x,\theta) = f(x,\theta)$ components $\mathcal{F}_\mu$ and $F_{\mu\nu}$ also are real, and propagating solutions do not exist for $m = 0$.

If for $s = 1/2$ and for $s = 1$ the first equation of the system (3.22)-(3.23) is the consequence of the second equation, then for $s > 1$ there are the solutions of equation (3.23) with mass spectrum, $m_i|s_i| = ms$, $s_i = s, s-1, \ldots, -s$. For the extraction of characterized by certain mass m and spin s representations of improper Poincaré group it is necessary to use both equations of the system.

Notice that the chirality λ of a particle described by (3.15)-(3.16) can be fractional, but the spin s of a particle described by (3.22)-(3.23) can be only integer or half-integer for $m \neq 0$ and finite number of components $\psi(x)$.

Really, if $2s$ is not integer, then acting by the raising operator on the state with label $\lambda = -s$, we not get into the state labelled by $\lambda = s$ and connected with initial state by the space reflection; moreover, the spectrum of λ is not bounded above. On the other hand, it is possible to develop an alternative approach (in particular, for the massive particles with fractional spin) based on the using of infinite-dimensional unitary irreps of $SO(2,1)$.

2. Let us consider now the solutions of (3.22)-(3.23) in the space of the square-integrable functions of θ. In the rest frame, as we have seen above, there are two types of the solutions. The solutions $[\sinh(\theta/2)]^{2s}$ are not square-integrable for any s since corresponding integral is divergent either at zero or at infinity. The solutions $[\cosh(\theta/2)]^{2s}$ for $s < 0$ are square-integrable:

$$\int_{-\infty}^{+\infty} [\cosh(\theta/2)]^{4s} d\theta = 2B(1/2,\, 2s).$$

Therefore in the space of square-integrable functions equation (3.23) has only positive energy solutions. Solutions with $p_0 < 0$ correspond to the equation $\hat{p}_\mu \hat{\Gamma}^\mu f(x,\theta) = -msf(x,\theta)$. Normalized positive energy solutions of the system (3.22)-(3.23) for the particle with spin $|s|$ and momenta $p_0 = m\cosh\phi$, $p_1 = m\sinh\phi$ are

$$f(x,\theta) = (2\pi)^{-1}\left(2B(1/2,\, 2s)\right)^{-1/2} e^{ip_0x^0 + ip_1x^1} \left(\cosh[(\theta+\phi)/2]\right)^{-2|s|}. \tag{3.30}$$

In contrast to the case $d > 2$ solutions with distinct s are nonorthogonal. The decomposition of the solutions (3.30) over the functions $e^{i\lambda\theta}$ (i.e. over $SO(1,1)$ unitary irreps) corresponds to the Fourier integral expansion. We will consider properties of the positive energy equations more detail in 2+1-dimensional case below.

IV. THREE-DIMENSIONAL CASE

A. Field on the group $M(3)$

The case of $M(3)$ group is characterized by many-dimensional spin space. On the other hand, the constructions allow the simple physical interpretation.

Using the operators $\hat{J}^i = \hat{L}^i + \hat{S}^i = (1/2)\epsilon^{ijk}\hat{J}_{jk}$, it is possible to rewrite the commutation relations (2.37) in the more compact form

$$[\hat{p}_i, \hat{p}_k] = 0, \quad [\hat{p}^i, \hat{J}^j] = i\epsilon^{ijk}\hat{p}_k, \quad [\hat{J}^i, \hat{J}^j] = i\epsilon^{ijk}\hat{J}_k\ . \tag{4.1}$$

The invariant measure on the group is given by the formulas

$$d\mu(x, \mathbf{z}) = Cd^3x\delta(|z_1|^2 + |z_2|^2 - 1)d^2z_1d^2z_2 = \frac{1}{16\pi^2}d^3x\sin\theta d\theta d\phi d\psi. \tag{4.2}$$
$$-\infty < x < +\infty, \quad 0 < \theta < \pi, \quad 0 < \phi < 2\pi, \quad -2\pi < \psi < 2\pi,$$

where $z_1 = \cos\frac{\theta}{2}e^{i(\psi-\phi)/2}$, $z_2 = i\sin\frac{\theta}{2}e^{i(\psi+\phi)/2}$ are the elements of the first column of matrix (2.46), $z^2 = -z_1$, $z^1 = z_2$, and θ, ϕ, ψ are the Euler angles. The spin projection operators acting in the space of the functions on the group $f(x, \mathbf{z})$ have the form

$$\begin{aligned}
\hat{S}_k &= \frac{1}{2}(z\sigma_k\partial_z - \overset{*}{z}\overset{*}{\sigma}_k\partial_{\overset{*}{z}}), \quad z = (z^1\ z^2),\ \partial_z = (\partial/\partial z^1\ \partial/\partial z^2)^T, \\
\hat{S}^R_k &= -\frac{1}{2}(\chi\overset{*}{\sigma}_k\partial_\chi - \overset{*}{\chi}\sigma_k\partial_{\overset{*}{\chi}}), \quad \chi = (z^1\ -\overset{*}{z}{}^{\dot{2}}),\ \partial_\chi = (\partial/\partial z^1\ -\partial/\partial\overset{*}{z}{}^{\dot{2}})^T.
\end{aligned} \tag{4.3}$$

In terms of Euler angles one can obtain

$$\hat{S}_3 = -i\partial/\partial\phi, \quad \hat{S}^R_3 = i\partial/\partial\psi. \tag{4.4}$$

The operator $\hat{\mathbf{p}}^2$ and the operator of the spin projection on the direction of propagation $\hat{W} = \hat{\mathbf{p}}\hat{\mathbf{J}} = \hat{\mathbf{p}}\hat{\mathbf{S}}$ are the Casimir operators. The eigenvalues $S(S+1)$ of the Casimir operator of rotation subgroup in z-space $\hat{\mathbf{S}}^2 = \hat{\mathbf{S}}^2_R$ define spin S. complete sets of the commuting operators $\{\hat{p}_k, \hat{W}, \hat{\mathbf{S}}^2, \hat{S}^3_R\}$, $\{\hat{\mathbf{p}}^2, \hat{W}, \hat{\mathbf{J}}^2, \hat{S}_3, \hat{\mathbf{S}}^2, \hat{S}^R_3\}$ consist of six operators (two Casimir operators, two left generators, and two right generators). The Casimir operator $\hat{W}$ does not commute with $\hat{L}_k$ and $\hat{S}_k$ separately but only with the generators $\hat{J}_k = \hat{L}_k + \hat{S}_k$, therefore there are sets, which do not include $\hat{W}$, for example, $\{\hat{\mathbf{p}}^2, \hat{p}_3, \hat{L}_3, \hat{S}_3, \hat{\mathbf{S}}^2, \hat{S}^R_3\}$ and $\{\hat{p}_\mu, \hat{S}_3, \hat{\mathbf{S}}^2, \hat{S}^R_3\}$.

We will consider the first set since in this case eigenfunctions have the most simple form. This set includes two Casimir operators, the operator of spin square $\hat{\mathbf{S}}^2$ and the generator $\hat{S}^R_3$. The latter two generators commute with all left generators but do not commute with right generators and label equivalent representations in the decomposition of the left GRR.

According to (4.4), the eigenfunctions of $\hat{S}^R_3$, $\hat{S}^R_3|\ldots n\rangle = n|\ldots n\rangle$, have the form $|\ldots n\rangle = F(x, \theta, \phi)\exp(-in\psi)$ and are differed only by a phase factor. As a consequence of

the commutation relations of generators $\hat{S}_k^R$ the operators $\hat{S}_\pm^R = \hat{S}_1^R \pm i\hat{S}_2^R$ are the raising and lowering operators for the eigenfunctions of $\hat{S}_3^R$

$$\hat{S}_\pm^R|\ldots n\rangle = C(S,n)|\ldots n \pm 1\rangle. \tag{4.5}$$

The intertwining operators $\hat{S}_\pm^R$ consist of the generators of right rotations, which conserve the interval square according to (2.35). Moreover, the right rotations do not act on x. But there are no transformations (rotations and translations) of the reference frame, which connect the representations with different n. Notice that the states labelled by n and $-n$ are interchanged under charge conjugation (2.63).

The operator $\hat{\mathbf{S}}^2$ also labels equivalent representations of $M(3)$ group. This operator commutes with all generators except right translations, and therefore an intertwining operator is a function of the latter generators. Right translations change both the interval and spin. Therefore it is naturally to characterize free particle in three-dimensional Euclidean space not only by momentum and spin projection on the direction of propagation, but also by spin S.

There are two standard realizations of the representation spaces, which correspond to eigenvalues $n = \pm 2S$ and $n = 0$ of the operator $\hat{S}_3^R$.

The first realization is the space of analytic ($n = -2S$) functions $f(x, z)$ or antianalytic ($n = 2S$) functions $f(x, \overset{*}{z})$ of two complex variables z^1, z^2, $|z^1|^2 + |z^2|^2 = 1$, i.e. the space of functions of two-component spinors. In particular, according to (4.3), for the space of analytic functions we have

$$\hat{S}_k = \frac{1}{2} z\sigma_k \partial_z, \tag{4.6}$$

$\hat{S}_3^R = -(z^1\partial/\partial z^1 + z^2\partial/\partial z^2)$, and $\hat{\mathbf{S}}^2 = \hat{S}_3^R(\hat{S}_3^R - 1)$. The eigenfunctions of the operator of spin square are polynomials of the power $2S$ in z^1, z^2. The charge conjugation transformation connects equivalent irreps labelled by $n = \pm 2S$ and the spaces of analytic and antianalytic function. This transformation reverses the direction of momentum and spin.

The second realization is the space of functions, which do not depend on the angle ψ, and corresponds to $n = 0$. It is the space of functions of five real variables on the manifold

$$\mathbb{R}^3 \times S^2, \qquad d\mu = (4\pi)^{-1} d^3x \sin\theta d\theta d\phi.$$

The point in the spin space (i.e. on the sphere $S^2 \sim \mathbb{C}P^1 \sim SU(2)/U(1)$) can be define by the spherical coordinates θ, ϕ, or by two complex variables $z_1 = \cos\frac{\theta}{2}e^{-i\phi/2}$, $z_2 = \sin\frac{\theta}{2}e^{i\phi/2}$ (in this case one may use (4.6) for the spin projection operators), or by one complex number $z = z_1/z_2$ (this case corresponds to the realization in terms of projective space $\mathbb{C}P^1$). In terms of variables θ, ϕ the eigenfunctions of operators $\hat{S}$, $\hat{S}_3$ are $P_S^s(\cos\theta)e^{is\phi}$, where $P_S^s(\cos\theta)$ are associated Legendre functions [14].

Let us consider eigenfunctions of the set of the operators $\{\hat{p}_\mu, \hat{W}, \hat{\mathbf{S}}^2\}$ in the space of analytic functions of z^1, z^2:

$$\hat{p}_\mu f(x,z) = p_\mu f(x,z), \quad \hat{\mathbf{S}}^2 f(x,z) = S(S+1)f(x,z), \quad \hat{\mathbf{p}}\hat{\mathbf{S}} f(x,z) = ps f(x,z). \tag{4.7}$$

The eigenfunctions of $\hat{\mathbf{S}}^2$ are polynomials of the power $2S$ in z (the unitary irreps of $SU(2)$ are finite-dimensional, therefore spin S and spin projection on the direction of propagation

s are integer or half-integer). Let $p_\mu = (0, 0, p)$, then the normalized solutions of the system (4.7) are

$$|0\,0\,p\,S\,s\rangle = (2\pi)^{-3/2}\left(\frac{(2S)!}{(S+s)!(S-s)!}\right)^{1/2}(z^1)^{S+s}(z^2)^{S-s}e^{ix_3p}.$$

The states with arbitrary direction of vector p may be obtain by the rotation $P = UP_0U^\dagger$, $Z = UZ_0$, $P_0 = p\sigma_3$, $Z_0 = (z_1\ z_2)^T$,

$$|p_1\,p_2\,p_3\,S\,s\rangle = (2\pi)^{-3/2}\left(\frac{(2S)!}{(S+s)!(S-s)!}\right)^{1/2}(z^1\overset{*}{u}_1 + z^2\overset{*}{u}_2)^{S+s}(-z^1u_2 + z^2u_1)^{S-s}e^{ipx}, \quad (4.8)$$

where u_1, u_2 are the elements of the first line of matrix U. Notice that it is sufficient to use only two angles for the parametrization of matrix U since the initial state has a stationary subgroup $U(1)$.

For the rest particle $\hat{\mathbf{p}}^2 = \hat{\mathbf{p}}\hat{\mathbf{S}} = 0$ and only in this case $M(3)$ irreps labelled by different S are nonequivalent.

In general case functions corresponding to the particle of spin S have the form

$$f_S(x,z) = \sum_{n=0}^{2S}\phi^n(z)\psi_n(x), \quad \phi^n(z) = (C^n_{2S})^{1/2}(z^1)^{S-n}(z^2)^n, \qquad (4.9)$$

$$\int \overset{*}{f}_S(x,z)f'_{S'}(x,z)d\mu(x,z) = \delta_{SS'}\int\sum_{n=0}^{2S}\overset{*}{\psi}_n(x)\psi'_n(x)d^3x, \qquad (4.10)$$

where C^{2S}_n is the binomial coefficient, and $d\mu(x,z)$ is the invariant measure (4.2). The relation (4.9) gives the connection between the description by the functions $f(x,z)$ and the standard description by the multicomponent functions $\psi_n(x)$. It is easy to see that the action of the operators $\hat{S}_k = \frac{1}{2}z\sigma_k\partial_z$ on the function (4.9) reduces to the multiplication of the column $\psi(x)$ by $(2S+1)\times(2S+1)$ matrices S_k of $SU(2)$ generators in the representation T_S, $\hat{S}_kf(x,z) = \phi(z)S_k\psi(x)$. Matrices S_k obey the commutation relations of spin projection operators, namely $[S^i, S^j] = i\epsilon^{ijk}S_k$.

In particular, the linear function of z^1,z^2 corresponds to spin $S = 1/2$, and the action of the operators $\hat{S}_k$ on $\psi(x)$ is reduced to the multiplication by σ-matrices, $\hat{S}_kf(x,z) = \phi(z)\sigma_k\psi(x)$.

As was mentioned above, the operator $\hat{\mathbf{S}}^2$ is not a Casimir operator of $M(3)$ and labels equivalent representations of the group. This operator is the direct analog of the Lorentz spin operator in pseudoeuclidean case, and we will consider its properties in details.

1. Operator $\hat{\mathbf{S}}^2$ is composed of right generators commuting with all left generators and therefore is not changed under the coordinate transformation (left transformations of the Euclidean group). The right transformations do not change the spin projection s on the direction of propagation but change both spin S and interval (distance).

2. Operator $\hat{\mathbf{S}}^2$ does not depend on x and commutes with operators $x_k, \hat{p}_k, \hat{S}_k$, therefore in the presence of interactions is conserved for any Hamiltonian $\hat{H} = \hat{H}(x_k, \hat{p}_k, \hat{S}_k)$.

3. The eigenvalues of $\hat{\mathbf{S}}^2$ label irreps of the rotation subgroup in the spin space and define the possible values of the spin projection s, which can arise under the interactions.

Notice that in the representation theory of Galilei group (symmetry group of nonrelativistic mechanics, which includes $M(3)$ and ensures more general description) irreps labelled

by different eigenvalues of $\hat{\mathbf{S}}^2$ are not equivalent. The classification of irreps of Galilei group can be based on the use of two invariant equations. The Schrödinger equation fixes the mass m, and the second equation fixes the eigenvalue of spin operator $\hat{\mathbf{S}}^2$ [82,83].

B. Field on the group $M(2,1)$ and fractional spin

Using the operators $\hat{J}^\rho = \hat{L}^\rho + \hat{S}^\rho = (1/2)\epsilon^{\rho\mu\nu}\hat{J}_{\mu\nu}$, it is possible to rewrite the commutation relations (2.37) in the next form:

$$[\hat{p}_\mu, \hat{p}_\nu] = 0, \quad [\hat{p}^\mu, \hat{J}^\nu] = -i\epsilon^{\mu\nu\eta}\hat{p}_\eta, \quad [\hat{J}^\mu, \hat{J}^\nu] = -i\epsilon^{\mu\nu\eta}\hat{J}_\eta\ . \tag{4.11}$$

The invariant measure on the group is given by the formulas [14]

$$d\mu(x,\underline{z}) = d\mu(z)d^3x = Cd^3x\delta(|z_1|^2 - |z_2|^2 - 1)d^2z_1d^2z_2 = \frac{1}{8\pi^2}d^3x\sinh\theta d\theta d\phi d\psi. \tag{4.12}$$
$$-\infty < x < +\infty, \quad 0 < \theta < \infty, \quad 0 < \phi < 2\pi, \quad -2\pi < \psi < 2\pi,$$

where $z_1 = \cosh\frac{\theta}{2}e^{i(\psi-\phi)/2}$, $z_2 = \sinh\frac{\theta}{2}e^{i(\psi+\phi)/2}$ are the elements of the first column of matrix Z (2.46), and θ, ϕ, ψ are the analogs of Euler angles, $z^2 = -z_1$, $z^1 = z_2$. The spin projection operators acting in the space of the functions on the group $f(x,\mathbf{z})$ have the form

$$\begin{aligned}
\hat{S}^\mu &= \frac{1}{2}(z\gamma^\mu\partial_z - \overset{*}{z}\overset{*}{\gamma}{}^\mu\partial_{\overset{*}{z}}), \quad z = (z^1\ z^2),\ \partial_z = (\partial/\partial z^1\ \partial/\partial z^2)^T,\\
\hat{S}^\mu_R &= -\frac{1}{2}(\chi\overset{*}{\gamma}{}^\mu\partial_\chi - \overset{*}{\chi}\gamma^\mu\partial_{\overset{*}{\chi}}), \quad \chi = (z^1\ \overset{*}{z}{}^{\dot{2}}),\ \partial_\chi = (\partial/\partial z^1\ \partial/\partial\overset{*}{z}{}^{\dot{2}})^T,
\end{aligned} \tag{4.13}$$

where γ^μ are three-dimensional γ-matrices,

$$\gamma^\mu = (\sigma_3, i\sigma_2, -i\sigma_1), \qquad \gamma^\mu\gamma^\nu = \eta^{\mu\nu} - i\varepsilon^{\mu\nu\rho}\gamma_\rho. \tag{4.14}$$

Notice that nonequivalent set of γ-matrices, $\gamma^\mu\gamma^\nu = \eta^{\mu\nu} + i\varepsilon^{\mu\nu\rho}\gamma_\rho$, is used in some papers. In terms of the Euler angles one may obtain $\hat{S}^0 = -i\partial/\partial\phi$, $\hat{S}^0_R = i\partial/\partial\psi$. The sets of commuting operators are the same as in Euclidean case.

In consequence of the identity $\sigma_1\overset{*}{U}\sigma_1 = U$ one may show that matrix σ_1 is the invariant symmetric tensor converting dotted and undotted indices,

$$\overset{*}{z}_\alpha = (\sigma_1)_\alpha{}^{\dot{\alpha}}\overset{*}{z}_{\dot{\alpha}}. \tag{4.15}$$

According to (2.47), the invariant tensor $\sigma_{\mu\alpha\dot{\alpha}}$ connects vector index and two spinor indices of different types. On the other hand, using the identity mentioned above one may rewrite (2.47) in the form $x'^\nu(\sigma_\mu\sigma_1) = x^\mu U(\sigma_\mu\sigma_1)U^T$. Thus the invariant tensor, which we denote as

$$\check{\sigma}_{\mu\alpha\beta} = (\sigma_\mu\sigma_1)_{\alpha\beta}, \quad \check{\sigma}_{\mu\alpha\beta} = \check{\sigma}_{\mu\beta\alpha}, \tag{4.16}$$

connects vector index and two spinor indices of one type. Thus, one can write the generators $\hat{S}^\mu$ in the form $\hat{S}^\mu = \frac{1}{2}\check{\sigma}^\mu{}_{\alpha\beta}(z^\alpha\partial^\beta + \overset{*}{z}{}^\alpha\overset{*}{\partial}{}^\beta)$. An analog of $\sigma^{\mu\nu}$-matrices in 2+1 dimensions is

$(\sigma^{\mu\nu})_{\alpha\beta} = \varepsilon^{\mu\nu\lambda}\check{\sigma}_{\lambda\alpha\beta}$. Raising one of the spinor indices of $\check{\sigma}_{\mu\alpha\beta}$, one may obtain two sets of three-dimensional γ-matrices differed only by signs of γ^0 and γ^2.

Similarly to the Euclidean case, there are two standard realizations of the representation spaces. These realizations correspond to eigenvalues $n = \pm 2S$ and $n = 0$ of the operator $\hat{S}_3^R$.

The first realization is the space of analytic ($n = -2S$) functions $f(x, z)$ or antianalytic ($n = 2S$) functions $f(x, \overset{*}{z})$ of two complex variables z^1, z^2, $|z^2|^2 - |z^1|^2 = 1$, i.e. the space of functions of two-component spinors. The eigenfunctions of $\hat{S}_\mu \hat{S}^\mu$ are homogeneous functions of degree $2S$ in z. According to (4.3), we have $\hat{S}^0_R = -(z^1\partial/\partial z^1 + z^2\partial/\partial z^2)$ for the space of analytic functions and $\hat{S}^0_R = \overset{*}{z}{}^{\dot{1}}\partial/\partial\overset{*}{z}{}^{\dot{1}} + \overset{*}{z}{}^{\dot{2}}\partial/\partial\overset{*}{z}{}^{\dot{2}}$ for the space of antianalytic functions. The eigenfunctions of $\hat{S}_\mu \hat{S}^\mu$ in these spaces are also eigenfunctions of $\hat{S}^0_R$ with eigenvalues $n = \mp 2S$ respectively.

The second realization is the space of eigenfunctions of $\hat{S}^0_R$ with zero eigenvalue. It is the space of functions of five real parameters on the manifold

$$\mathbb{R}^3 \times \mathbb{C}D^1, \qquad d\mu = (2\pi)^{-1} d^3x \sinh\theta d\theta d\phi,$$

where $\mathbb{C}D^1 \sim SU(1,1)/U(1)$ is a complex disk. This functions do not depend on the angle ψ.

Remember some facts from the representation theory of $SU(1,1)$. For finite-dimensional nonunitary irreps T^0_S of 2+1 Lorentz group $SU(1,1) \sim SO(2,1)$ spin projection s (the eigenvalue of $\hat{S}^0$) can be only integer or half-integer, $s = -S, \dots, S$, where $S \geq 0$. However, in 2+1 dimensions Lorentz group has not compact non-Abelian subgroup. Therefore there are infinite-dimensional unitary representations corresponding to fractional S. These representations are multi-valued representations of $SU(1,1)$. For single-valued representations of $SU(1,1)$ the spin projection s can be only integer or half-integer (for $SO(2,1)$ only integer).

The representations of discrete seria correspond to $S < -1/2$. Irreps of the positive discrete series T^+_S are bounded by lowest weight $s = -S$, irreps of the negative discrete series T^-_S are bounded by highest weight $s = S$, irreps of the principal series correspond to $S = -1/2 + i\lambda$ and can be bounded by highest (lowest) weight only for $S = -1/2$. For other irreps of principal series the spectrum of s is not bounded. Supplementary series correspond to $-1/2 < S < 0$ and are characterized by nonlocal scalar product.

A visual picture for weight diagrams of all seria on the plain S, s one can find in [30,84].

Thus there are only two possibilities for description of a particle with fractional spin by means of unitary irreps of $SU(1,1)$ with local scalar product. The first corresponds to the discrete or principal seria irreps bounded by lowest (highest) weight, $|s| \geq |S| \geq 1/2$. The second corresponds to the principal series irreps which is not bounded.

Unitary irreps of discrete seria are used for the description of anyons [42,44,85,30]. In [42,44,85] corresponding unitary infinite-component representations of $M(2,1)$ were constructed in the space of functions of x^μ and complex variable $z = z^1/z^2$, i.e. on the coset space $M(2,1)/U(1)$. It was shown that RWE associated with irreps of the discrete seria have solutions only with definite sign of energy. Thus mentioned RWE are analogs of Majorana equations in 3+1 dimensions; this aspect was considered more detail in [85]. Various formulations of the higher spin theory based on finite-component representations were considered, in particular, in [86–90].

C. Relativistic wave equations in 2+1 dimensions

Let us fix eigenvalues of the Casimir operators of the Poincaré group and of spin Lorentz subgroup:

$$\hat{p}^2 f(x,\mathbf{z}) = m^2 f(x,\mathbf{z}), \tag{4.17}$$
$$\hat{p}_\mu \hat{S}^\mu f(x,\mathbf{z}) = K f(x,\mathbf{z}), \tag{4.18}$$
$$\hat{S}_\mu \hat{S}^\mu f(x,\mathbf{z}) = S(S+1) f(x,\mathbf{z}). \tag{4.19}$$

Below we will call the operator $\hat{S}_\mu \hat{S}^\mu$ as operator of the Lorentz spin square.

Equations (4.17),(4.18) define some sub-representation of the left GRR of $M(2,1)$, which is characterized by mass m, Lorentz spin S, and by the eigenvalue K of Lubanski-Pauli operator. At $m=0$ we suppose $K=0$, that is true for irreps with finite number of spinning degrees of freedom. The general cases for $m=0$ and for m imaginary were discussed in [91,30].

Possible values of K can be easily described in the massive case. Here we can use a rest frame, where $\hat{p}_\mu \hat{S}^\mu = \hat{S}^0 m \,\mathrm{sign}\, p_0$. Thus, $K = sm = s^0 m$ for $p_0 > 0$ and $K = sm = -s^0 m$ for $p_0 < 0$, where s^0 is the eigenvalue of $\hat{S}^0$. The spectrum of $\hat{S}^0$ depends on the representation of the Lorentz group.

Variable s labels irreps of the group $M(2,1)$ and can take both positive and negative values. Thus there exist the analogy with characterized by helicity massless particles in 3+1 dimensions. In both cases $SO(2)$ is the little group, and single-valued irreps of $SO(2)$ are labelled by integer number $2s$. (It is a particular case of the connection between the massive fields in d dimensions and massless fields in $d+1$ dimensions, see [92,90]). Therefore we will call s as helicity and $|s|$ as spin.

Corresponding to (2.61), space reflection reduces to the rotation at π around axis x^0 and converts Z to $(Z^\dagger)^{-1} = \sigma_3 Z \sigma_3$, or $z_1 \to z_1$, $z_2 \to -z_2$. Operators $\hat{p}^0$, $\hat{S}^0$ do not change. Thus, distinct from 3+1-dimensional case, space reflection leaves helicity unaltered.

Fixing S in (4.19), we pass to the space of homogeneous functions of degree $2S$ in z_1, z_2. According to the sign of S, below we will consider two possible choices of $SU(1,1)$ irreps bounded either with both sides or with one side respectively.

Finite-dimensional nonunitary irreps T^0_S of $SU(1,1)$ are labelled by positive integer or half-integer S. The basis in the representation space is formed by the polynomials of power $2S$ in z, see (A2). We denote corresponding representations of $M(2,1)$ as $T^0_{m,s}$.

Infinite-dimensional unitary irreps T^-_S (T^+_S) of $SU(1,1)$ are labelled by negative $S < -1/2$ and are bounded by highest (lowest) weight. The basis in the representation space is formed by the quasipolynomials of power $2S$ in z, see (A3). We denote corresponding representations of $M(2,1)$ as $T^-_{m,s}$ ($T^+_{m,s}$).

One may present a function $f(x,z)$ in the form

$$f(x,z) = \phi(z)\psi(x), \tag{4.20}$$

where $\phi(z)$ is a line composed of the basis elements $\phi_n(z)$ of the corresponding $SU(1,1)$ representation, and $\psi(x)$ is a column composed of the coefficients in the decomposition over this basis. The action of the differential operators $\hat{S}^\mu$ on a function $f(x,z)$ may be presented in terms of matrices

$$\hat{S}^\mu f(x,z) = \phi^n(z)(S^\mu)_n{}^{n'}\psi_{n'}(x), \tag{4.21}$$

where S^μ are $SU(1,1)$ generators in the representation T_S (see Appendix and also [30]). They obey the commutation relations of the $SU(1,1)$ group $[S^\mu, S^\nu] = -i\epsilon^{\mu\nu\eta}S_\eta$.

For fixed S in the matrix representation equations (4.17),(4.18) have the form

$$(\hat{p}^2 - m^2)\psi(x) = 0, \tag{4.22}$$

$$(\hat{p}_\mu S^\mu - sm)\psi(x) = 0, \tag{4.23}$$

According to (4.23),

$$\psi^\dagger(x)(S^{\dagger\mu}\overleftarrow{\hat{p}_\mu} + sm) = 0.$$

It follows from the explicit expressions (A4) that for $T^0_{m,s}$ the relation $S^{\dagger\mu} = \Gamma S^\mu \Gamma$, where relations $(\Gamma)_{nn'} = (-1)^n\delta_{nn'}$, $\Gamma^2 = 1$ take place. For $T^+_{m,s}$ and $T^-_{m,s}$ matrices S^μ are Hermitian, $S^{\dagger\mu} = S^\mu$, according to (A5). Let us introduce the notation

$$\overline{\psi} = \psi^\dagger\Gamma \quad \text{for} \quad T^0_{m,s},$$
$$\overline{\psi} = \psi^\dagger \quad \text{for} \quad T^+_{m,s}, T^-_{m,s}.$$

The function $\overline{\psi}(x)$ obeys the equation

$$\overline{\psi}(x)(S^\mu\overleftarrow{\hat{p}_\mu} + sm) = 0. \tag{4.24}$$

As a consequence of the relations $S^{\dagger\mu} = \Gamma S^\mu \Gamma$ and $(S^\mu)^\dagger = -(-1)^{\delta_{0\mu}}S^\mu$ we obtain that for irrep $T^0_{m,s}$ finite transformation matrices obey the equation $\Gamma T^\dagger(g)\Gamma = T^{-1}(g)$. Therefore $\overline{\psi}(x)\psi(x)$ is a scalar density, and one may define a scalar product in the space of columns

$$(\psi'(x), \psi(x)) = \int \overline{\psi}'(x)\psi(x)d^3x. \tag{4.25}$$

The scalar density is positive definite for $T^+_{m,s}$ and $T^-_{m,s}$ in contrast to the case of $T^0_{m,s}$.

As a consequence of (4.23) and (4.24), the continuity equation holds

$$\partial_\mu j^\mu = 0, \quad j^\mu = \overline{\psi}S^\mu\psi. \tag{4.26}$$

Together with the current vector j^μ, by analogy with four-dimensional case [59], one can associate with the linear equation (4.23) the energy-momentum tensor $T^{\mu\nu}$ and the energy density $W = -T^{00}$:

$$T^{\mu\nu} = \mathrm{Im}\left(S^\mu\frac{\partial\psi}{\partial x^\nu}, \psi\right), \qquad W = -T^{00} = -\mathrm{Im}\left(S^0\frac{\partial\psi}{\partial x^0}, \psi\right). \tag{4.27}$$

If matrix S^0 is diagonable, then the positiveness of $W(x)$ is equivalent to the requirement that

$$(S^0\psi, S^0\psi) \geq 0 \tag{4.28}$$

for all vectors ψ [59]. In particular, for $T^+_{m,s}$ and $T^-_{m,s}$ the relation $(S^0\psi, S^0\psi) = \psi^\dagger S^0 S^0 \psi \geq 0$ takes place, and energy density is positive definite.

There are two cases when equation (4.22) is the consequence of (4.23). Indeed, multiplying equation (4.18) by $\hat{p}_\mu S^\mu + ms$, one gets

$$(\hat{p}_\mu S^\mu + ms)(\hat{p}_\nu S^\nu - ms)\psi(x) = \left(\tfrac{1}{2}\hat{p}_\mu \hat{p}_\nu [S^\mu,\ S^\nu]_+ - m^2 s^2\right)\psi(x) = 0. \qquad (4.29)$$

In the particular case $S = 1/2$ we have $s = \pm 1/2$, $S^\mu = \gamma^\mu/2$, and (4.29) is merely the Klein-Gordon equation (4.22). In general case the matrices S^μ are not γ-matrices in higher dimensions, and the squared equation (4.29) does not coincide with the Klein-Gordon equation (4.22). Using the rest frame, one may show that the equation (4.22) follows from (4.23) also in the case of vector representation $S = 1$, $s = \pm 1$. In the other cases for the identification of the irrep of $M(2,1)$ it is necessary to use both equations of the system (4.22),(4.23). Notice that another approach to the description of fields with fixed spin and mass was suggested in [93]; this approach is based on the system of spinor linear equations.

It is naturally to connect spin value with the highest (lowest) weight of the irrep of Lorentz group, $s = \pm S$. This means that up to a sign ($+$ for $p_0 > 0$, $-$ for $p_0 < 0$) s is equal to maximal or minimal eigenvalue of the operator $\hat{S}^0$ in the representation T_S of the Lorentz group. According to (4.17)-(4.19), in this case functions $f(x,z)$ obey the equations

$$\hat{p}^2 f(x,\mathbf{z}) = m^2 f(x,\mathbf{z}), \qquad (4.30)$$
$$\hat{p}_\mu \hat{S}^\mu f(x,\mathbf{z}) = msf(x,\mathbf{z}), \quad s = \pm S, \qquad (4.31)$$
$$\hat{S}_\mu \hat{S}^\mu f(x,\mathbf{z}) = S(S+1) f(x,\mathbf{z}). \qquad (4.32)$$

In the framework of the system (4.30)-(4.32) *there are two possibilities to describe one and the same spin*:

1. Equations for $f(x,\mathbf{z}) = \phi(\mathbf{z})\psi(x)$, where $\phi(\mathbf{z})$ transform under finite-dimensional nonunitary irrep of the Lorentz group.

2. Equations for $f(x,\mathbf{z}) = \phi(\mathbf{z})\psi(x)$, where $\phi(\mathbf{z})$ transform under infinite-dimensional unitary irrep of the Lorentz group. These equations allow us to describe also particles with fractional spin (anyons).

(1) At first, consider the Poincaré group representations $T^0_{m,s}$ associated with *finite-dimensional non-unitary irreps* of $SU(1,1)$. In this case S has to be positive, integer or half-integer. In the rest frame the solutions of the system (4.30)-(4.32) in the space of analytic functions (polynomials of power $2S$ in z^1, z^2) are

$$s = S > 0: \quad f(x,z) = C_1 (z^1)^S e^{imx^0} + C_2 (z^2)^S e^{-imx^0}, \qquad (4.33)$$
$$s = -S < 0: \quad f(x,z) = C_3 (z^1)^S e^{-imx^0} + C_4 (z^2)^S e^{imx^0}. \qquad (4.34)$$

For unique mass and spin there exist four independent components differed by signs of p_0 and s, which correspond to four irreps of $M(2,1)$. The separation by the sign of helicity s has absolute character since these states are solutions of different equations. But the states with different sign of p_0 are solutions of one and the same equation. Hence, the energy spectrum of solutions is not bounded below or above.

In the space of antianalytic functions (polynomials of power $2S$ in $\overset{*}{z}{}^{\dot{1}}$, $\overset{*}{z}{}^{\dot{2}}$) solutions of the system (4.30)-(4.32) are

$$
\begin{aligned}
s = S > 0: &\quad f(x, \overset{*}{z}) = C_1(\overset{*}{z}{}^{\dot{1}})^S e^{-imx^0} + C_2(\overset{*}{z}{}^{\dot{2}})^S e^{imx^0}, \\
s = -S < 0: &\quad f(x, \overset{*}{z}) = C_3(\overset{*}{z}{}^{\dot{1}})^S e^{imx^0} + C_4(\overset{*}{z}{}^{\dot{2}})^S e^{-imx^0}.
\end{aligned}
$$

These solutions are connected with previous case (4.33),(4.34) by charge conjugation (2.63) and therefore may be treated as the solutions describing antiparticles.

The wave function (4.33) corresponding to the helicity $s = -S$ has the form $C(z^2)^{2S} e^{ip_0 x^0}$, $p_0 = m$, in the rest frame. Acting on it by finite transformations, we get a solution in the form of the plane wave, which is characterized by the momentum p:

$$
\begin{aligned}
&P = U P_0 U^\dagger, \quad P_0 = mI, \quad Z = U Z_0, \quad Z_0 = (z_1\, z_2)^T, \\
&f(x, z) = (2\pi)^{-3/2} \left(z^2 u_1 - z^1 u_2\right)^{2S} e^{ipx}.
\end{aligned} \tag{4.35}
$$

The state with $P_0 = mI$ has the stationary subgroup $U(1)$, and we can take elements $u_1 = \cosh\theta/2$ and $u_2 = \sinh\theta/2\, e^{i\omega}$ of the first line of matrix U, that depends on two parameters only. Thus $p_0 = E = m\cosh\theta$, $\quad -p_1 + ip_2 = m\sinh\theta e^{i\omega}$, and one can express the parameters u_1 and u_2 via the momentum p:

$$
\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{1}{\sqrt{2m(E+m)}} \begin{pmatrix} E + m \\ -p_1 + ip_2 \end{pmatrix}. \tag{4.36}
$$

$2S+1$ components $\psi_n(x)$ are the coefficients in the decomposition of the function (4.35) over the basis $\phi^n(z)$, $f(x,z) = \phi^n(z)\psi_n(x)$, $\quad n = 0, 1, \ldots, 2S$:

$$
\begin{aligned}
\psi_n(x) &= (2\pi)^{-3/2} \left(C_{2S}^n\right)^{1/2} (u_1)^{2S-n} (-u_2)^n e^{ipx} \\
&= (2\pi)^{-3/2} \left(C_{2S}^n\right)^{1/2} \frac{(E+m)^{2S-n} (p_1 - ip_2)^n}{(2m(E+m))^S} e^{ipx}.
\end{aligned} \tag{4.37}
$$

In the particular case $S = 1/2$ we get

$$
\psi(x) = \frac{1}{\sqrt{2m(E-m)}} \begin{pmatrix} p_2 - ip_1 \\ E + m \end{pmatrix} e^{ipx}.
$$

Considering the system (4.31)-(4.32) without the condition of the mass irreducibility (4.30), it is easy to see that the charge density $j^0 = \psi^\dagger \Gamma S^0 \psi$ is positive definite only for $S = 1/2$, and the energy density $-T^{00}$ is positive definite only for $S = 1$. The scalar density $\overline{\psi}\psi = \psi^\dagger \Gamma \psi$ is not positive definite.

Let us show that for the particles with half-integer spin described by the system (4.30)-(4.32) the charge density j^0 (4.26) is positive definite. In the rest frame solutions of the system (4.30)-(4.32) have only two components (labelled by $s_0 = \pm S$), which we denote as $\psi_S(x)$ and $\psi_{-S}(x)$. For half-integer spin an inequality $j^0 = \psi^\dagger \Gamma S^0 \psi = S(|\psi_S|^2 + |\psi_{-S}|^2) > 0$ holds. At $S \geq 3/2$ from the explicit form of matrices S^1 and S^2 (A4) one can obtain that in the rest frame $j^1 = j^2 = 0$, therefore the square of the current vector $(j^0)^2 - (j^1)^2 - (j^2)^2$ is positive. Therefore $j^0 > 0$ for any plane wave.

Thus the charge density j^0 is positive definite for half-integer spin particles described by representations $T^0_{m,s}$ of $M(2,1)$. The scalar density and the energy density are proportional to $\psi^\dagger \Gamma \psi = |\psi_S|^2 - |\psi_{-S}|^2$ in the rest frame and therefore are indefinite.

Let us consider now particles with integer spin. In the rest frame solutions of the system also have only two components: $\psi_S(x)$ and $\psi_{-S}(x)$, $(S^0\psi, S^0\psi) = \psi^\dagger\Gamma S^0 S^0\psi = S^2(|\psi_S|^2 + |\psi_{-S}|^2) > 0$. Thus the energy density is positive definite for integer spin particles described by representations $T^0_{m,s}$ of $M(2,1)$. The charge density is proportional to $|\psi_S|^2 - |\psi_{-S}|^2$ in the rest frame and therefore is indefinite.

Consider two particular cases explicitly. For $S = 1/2$ the decomposition (4.20) has the following form

$$f(x,z) = z^1\psi_1(x) + z^2\psi_2(x), \quad \psi'(x') = U^{-1}\psi(x), \quad \psi(x) = (\psi_1(x)\ \psi_2(x))^T. \tag{4.38}$$

Taking into account the relation $U^{-1} = \sigma_3 U^\dagger \sigma_3$, which is valued for the $SU(1,1)$ matrices, we get the transformation law for the line $\bar\psi = \psi^\dagger\sigma_3$, $\bar\psi'(x') = \bar\psi(x)U$. The product $\bar\psi(x)\psi(x) = |\psi_1(x)|^2 - |\psi_2(x)|^2$ is the scalar density.

Thus, in the case under consideration, we have two equivalent descriptions. One in terms of functions (4.38) and another one in terms of lines $\bar\psi(x)$ or columns $\psi(x)$. One can find the action of the operators $\hat S^\mu$ in the latter representation, and equation (4.23) can be rewritten in the form of $2+1$ Dirac equation

$$\hat S^\mu\psi(x) = \frac{1}{2}\gamma^\mu\psi(x), \quad (\hat p_\mu\gamma^\mu \mp m)\psi(x) = 0, \tag{4.39}$$

where minus corresponds to $s = 1/2$, plus corresponds to $s = -1/2$, and γ^μ are 2×2 γ-matrices (4.14) in $2+1$ dimensions. The functions $\psi = (\psi^1\ 0)^T$ and $\psi = (0\ \psi^2)^T$ are eigenvectors of the operator $\hat S^0$ with the eigenvalues $\pm 1/2$.

Sometimes two equations for $s = \pm 1/2$ are written as one equation for the four-component reducible representation [89,94], $(\hat p_\mu\Gamma^\mu - m)\Psi(x) = 0$, where $\Gamma^\mu = \mathrm{diag}(\gamma^\mu,\ -\gamma^\mu)$, that corresponds to the simultaneous consideration of particles with opposite helicities.

For $S = 1$ the decomposition (4.20) has the following form

$$f(x,z) = \psi_{11}(x)(z^1)^2 + \psi_{12}(x)z^1z^2 + \psi_{22}(x)(z^2)^2, \tag{4.40}$$

where $\psi(x) = (\psi_{11}(x)\ \psi_{12}(x)/\sqrt{2}\ \psi_{22}(x))^T$ is subjected to equation (4.23) with the matrices

$$S^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad S^1 = -\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S^2 = -\frac{i}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{4.41}$$

If instead of the cyclic components $\psi_{\alpha\beta}(x)$ one introduces new (Cartesian) components $\mathcal{F}_\mu = \check\sigma_\mu{}^{\alpha\beta}\psi_{\alpha\beta}(x)$, where $\check\sigma_{\mu\alpha\beta}$ is defined in (4.16), $\mathcal{F}_0 = -2\psi^{12}$, $\mathcal{F}_1 = \psi^{11} + \psi^{22}$, $\mathcal{F}_2 = i(\psi^{22} - \psi^{11})$, then equation (4.23) takes the form [30]

$$\partial_\mu\varepsilon^{\mu\nu\eta}\mathcal{F}_\eta - sm\mathcal{F}^\nu = 0. \tag{4.42}$$

A transversality condition follows from (4.42), $\partial_\mu\mathcal{F}^\mu = 0$. One can see now that equations (4.42) are in fact field equations of the so called "self-dual" free massive field theory [95]. As remarked in [96], this theory is equivalent to the topologically massive gauge theory with the Chern-Simons term [97]. Indeed, the transversality condition allows introducing gauge potentials A_μ, namely a transverse vector can be written as a curl $\mathcal{F}^\mu = \varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda =$

$\varepsilon^{\mu\nu\lambda}F_{\nu\lambda}/2$, where $F_{\nu\lambda} = \partial_\nu A_\lambda - \partial_\lambda A_\nu$ is the field strength. Thus, $\mathcal{F}^\mu$ appears to be dual field strength, which is a tree-component vector in 2+1 dimensions. Then (4.42) implies the following equations for $F_{\mu\nu}$

$$\partial_\mu F^{\mu\nu} - \frac{sm}{2}\varepsilon^{\nu\alpha\beta}F_{\alpha\beta} = 0,$$

which are the field equations of the topologically massive gauge theory.

To describe neutral spin 1 particle coinciding with its antiparticle one may consider a function

$$f(x,\mathbf{z}) = \psi_{11}(x)z^1\overset{*}{z}{}^1 + \psi_{12}(x)(z^1\overset{*}{z}{}^2 + \overset{*}{z}{}^1 z^2)/2 + \psi_{22}(x)z^2\overset{*}{z}{}^2, \tag{4.43}$$

where we have used (4.15) for the convertation to undotted indices. The spin part of the function (4.43) depends not on three angles as in the case (4.40), but on two angles only. This function is an eigenfunction of operator $\hat{S}^3_R$ with zero eigenvalue. Substituting (4.43) into (4.31), we again obtain equation (4.42).

(2) Consider now Poincaré group representations $T^+_{m,s}$ and $T^-_{m,s}$ associated with *unitary infinite-dimensional irreps* of $SU(1,1)$ with highest (lowest) weight. In this case S can be non-integer, $S < -1/2$ (discrete series) or $S = -1/2$ (principal series). Eigenvalues of $\hat{S}^0$ can take only positive values for discrete positive series, $s^0 = -S + n$, and only negative valuses for negative one, $s^0 = S - n$, where $n = 0, 1, 2, \ldots$.

Let us consider the energy spectrum of the system (4.30)-(4.32) at $m \neq 0$. According to the first equation $p_0 = \pm m$. The second equation ensures the relation between spectra of the operators $\hat{p}_0$ and $\hat{S}^0$,

$$p_0 s^0 = ms. \tag{4.44}$$

For representations $T^0_{m,s}$, which correspond to finite-dimensional irreps T^0_S of the Lorentz group, the value of s^0 can be both positive and negative. Therefore for any s there exist both positive-frequency and negative-frequency solutions, and the representations $T^0_{m,s}$ splits into two irreps, characterized by $\operatorname{sign} p_0 = \pm 1$.

For unitary $SU(1,1)$ irreps with highest (lowest) weight the spectrum of $\hat{S}^0$ has definite sign. For T^+_S, which act in the space of analytic functions, the spectrum of operator $\hat{S}^0$ is positive, and for T^-_S, which act in the space of antianalytic functions, is negative. Therefore the sign of energy p_0 coincides with the sign of s for T^+_S, and the signs of p_0 and s are opposite for T^-_S. Thus $T^+_{m,s}$ and $T^-_{m,s}$ are irreps of $M(2,1)$.

As well as in the case of representations $T^0_{m,s}$, for unique mass and spin there are four states distinguished in signs of p_0 and s. In the rest frame there are two solutions of the system in the space of analytic functions:

$$p_0 > 0, s > 0: \quad f(x,z) = (2\pi)^{-3/2}(z^2)^S e^{imx^0}, \tag{4.45}$$

$$p_0 < 0, s < 0: \quad f(x,z) = (2\pi)^{-3/2}(z^2)^S e^{-imx^0}. \tag{4.46}$$

The solutions are connected by time reflection T' (2.62). In the space of antianalytic functions there are also two solutions:

$$p_0 > 0, s < 0: \quad f(x,\overset{*}{z}) = (2\pi)^{-3/2}(\overset{*}{z}{}^{\dot{2}})^S e^{imx^0} \tag{4.47}$$

$$p_0 < 0, s > 0: \quad f(x,\overset{*}{z}) = (2\pi)^{-3/2}(\overset{*}{z}{}^{\dot{2}})^S e^{-imx^0}. \tag{4.48}$$

These solutions are connected respectively with (4.45),(4.46) by Schwinger time reversal $T_{sch} = CT'$, which turns particles into antiparticles. Thus, there exist four equations distinguished in sign of s and by the used functional space (irrep T_S^+ or T_S^- of the Lorentz group), and any equation has the solutions only with definite sign of p_0.

In contrast to the case of $T^0_{m,s}$, where the energy spectrum p_0 is not bounded both above and below, the energy spectrum has definite sign. In any inertial frame the spectrum is bounded below by $p_0 = m$ for the solutions (4.45), (4.47) and above by $p_0 = -m$ for the solutions (4.46), (4.48).

For the unitary irreps of $M(2,1)$ under consideration, which correspond to the irreps of the discrete seria of the Lorentz group, integration of the functions (A3) in the invariant measure (4.12) gives

$$\int \overset{*}{f}_{S_1}(x,\mathbf{z}) f'_{S_2}(x,\mathbf{z}) d\mu(x,\mathbf{z}) = \delta_{S_1 S_2} \int \sum_{n=0}^{\infty} \psi^n(x) \psi'^n(x) d^3x, \tag{4.49}$$
$$\int \overset{*}{f}_{S_1}(x,\mathbf{z}) f'_{S_2}(x,\mathbf{z}) d\mu(\mathbf{z}) = \delta_{S_1 S_2} \psi^\dagger(x) \psi'(x),$$

In particular, the states (4.45)-(4.48) have the norm $\delta_{SS'}\delta(p-p')$. For the principal series $j = -1/2 + i\lambda$, and $\delta_{j_1 j_2}$ in (4.49) is changed by $\delta(\lambda_1 - \lambda_2)$. At the same time, the integral over the spin space diverges for the representations $T^0_{m,s}$, which correspond to finite-dimensional irreps of the Lorentz group.

Arbitrary plain wave solutions can be obtained by analogy with considered above case of $T^0_{m,s}$. For example, for the states (4.45) one can get the formula (4.37), where now C^n_{2S} are the coefficients from (A3) and $n = 0, 1, 2, \ldots$. The power $2S$ is negative, and the decomposition $f(x,z) = \phi_n(z)\psi^n(x)$ contains infinite number of terms.

Let us summarize some properties of the unitary irreps under consideration. Irreps $T^+_{m,s}$ and $T^-_{m,s}$ of the Poincaré group describe particles and antiparticles respectively. Charge density $j^0 = \psi^\dagger S^0 \psi$ is positive definite for particles and negative definite for antiparticles. The energy density is positive definite in both cases since $(S^0\psi, S^0\psi) = \psi^\dagger S^0 S^0 \psi > 0$. Besides, for the unitary irreps the scalar density $\psi^\dagger\psi$ is also positive definite in contrast to the finite-dimensional case. The existence of positive definite scalar density ensures the possibility of probability amplitude interpretation of $\psi(x)$.

Thus in 2+1 dimensions the problem of the construction of *positive-energy RWEs* is solved by the system (4.30)-(4.32) for the infinite-dimensional unitary irreps $T^+_{m,s}$ (signs of p_0 and s are the same) or $T^-_{m,s}$ (signs of p_0 and s are opposite) characterized by the mass m and the helicity s. These irreps of the Poincaré group are associated with irreps T_S^+ and T_S^- of the Lorentz group with lowest (highest) weight. Charge conjugation, changing signs of p_0 and s^0, leaves the helicity s invariant and turns $T^+_{m,s}$ into $T^-_{m,s}$.

An interesting problem is to find an explicit form of the intertwining operator A for the unitary irreps $T^+_{m,s}$, $T^-_{m,s}$ and the representation $T^0_{m,s}$ labelled by the same mass m and helicity s but assotiated with finite-dimensional nonunitary irreps of the Lorentz group, $AT^0_{m,s} = (T^+_{m,s} \oplus T^-_{m,s})A$. The intertwining operator is nonunitary and must be a function of the generators of right translations, since other generators commute with Lorentz spin square operator $\hat{S}_\mu \hat{S}^\mu$ and can't change the representation of spin Lorentz subgroup.

Notice that the 2+1 Dirac equation arises also in the case of unitary infinite-dimensional irreps T_S^+ and T_S^- of the Lorentz group not as an equation for a true wave function, but as

an equation for spin coherent states evolution. In this case the equation includes effective mass $m_s = |\frac{s}{S}|m$, $s = -S, -S+1, \dots$ [30].

Among the above considered RWE there exist ones which describe particles with fractional real spin. These equations are associated with unitary multi-valued irreps of the Lorentz group and can be used to describe anyons.

In spite of the fact that the number of independent polarization states for massive 2+1 particle is one, the vectors of the corresponding representation space of irreps $T^+_{m,s}$, $T^-_{m,s}$ have infinite number of components in matrix representation. Thus, z-representation is more convenient in this case.

V. FOUR-DIMENSIONAL CASE

A. Field on the group $M(3,1)$

The generators and the action of the left GRR on the functions $f(x, \mathbf{z})$ are given by formulas (2.37), (2.44). For spin projection operators it is convenient to use three-dimensional vector notation $\hat{S}_k = \frac{1}{2}\epsilon_{ijk}\hat{S}^{ij}$, $\hat{B}_k = \hat{S}_{0k}$. The explicit calculation gives

$$
\begin{aligned}
&\hat{S}_k = \frac{1}{2}(z\sigma_k\partial_z - \overset{*}{z}\overset{*}{\sigma}_k\partial_{\overset{*}{z}}) + \dots\,, \\
&\hat{B}_k = \frac{i}{2}(z\sigma_k\partial_z + \overset{*}{z}\overset{*}{\sigma}_k\partial_{\overset{*}{z}}) + \dots\,, \quad z = (z^1\ z^2), \quad \partial_z = (\partial/\partial z^1\ \partial/\partial z^2)^T; && (5.1) \\
&\hat{S}^R_k = -\frac{1}{2}(\chi\overset{*}{\sigma}_k\partial_\chi - \overset{*}{\chi}\sigma_k\partial_{\overset{*}{\chi}}) + \dots\,, \\
&\hat{B}^R_k = -\frac{i}{2}(\chi\overset{*}{\sigma}_k\partial_\chi + \overset{*}{\chi}\sigma_k\partial_{\overset{*}{\chi}}) + \dots\,, \quad \chi = (z^1\ \underline{z}^1), \quad \partial_\chi = (\partial/\partial z^1\ \partial/\partial \underline{z}^1)^T; && (5.2)
\end{aligned}
$$

Dots in the formulas replace analogous expressions obtaining by the substitutions $z \to z' = (\underline{z}^1\ \underline{z}^2)$, $\chi \to \chi' = (z^2\ \underline{z}^2)$.

Since $\det Z = 1$, then any of z_α, $\underline{z}_\alpha$ can be expressed in terms of other three, for example $\underline{z}_2 = (1 - z_2\underline{z}_1)/z_1$. Invariant measure on $\mathbb{R}^4 \times SL(2, C)$ has the form [46]

$$
d\mu(x, \mathbf{z}) = (i/2)^3 d^4x d^2z_1 d^2z_2 d^2\underline{z}_1 |z_1|^{-2}. \tag{5.3}
$$

The functions on the Poincaré group depend on 10 parameters, and correspondingly there are 10 commuting operators (two Casimir operators, four left and four right generators).

Both the Poincaré group and the spin Lorentz subgroup have two Casimir operators:

$$
\hat{p}^2 = \hat{p}_\mu\hat{p}^\mu, \quad \hat{W}^2 = \hat{W}_\mu\hat{W}^\mu, \quad \text{where} \quad \hat{W}^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\hat{p}_\nu\hat{J}_{\rho\sigma} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\hat{p}_\nu\hat{S}_{\rho\sigma}, \tag{5.4}
$$

$$
\frac{1}{2}\hat{S}_{\mu\nu}\hat{S}^{\mu\nu} = \frac{1}{2}\hat{S}^R_{\mu\nu}\hat{S}^{\mu\nu}_R = \hat{\mathbf{S}}^2 - \hat{\mathbf{B}}^2, \quad \frac{1}{16}\epsilon^{\mu\nu\rho\sigma}\hat{S}_{\mu\nu}\hat{S}_{\rho\sigma} = \frac{1}{16}\epsilon^{\mu\nu\rho\sigma}\hat{S}^R_{\mu\nu}\hat{S}^R_{\rho\sigma} = \hat{\mathbf{S}}\hat{\mathbf{B}}. \tag{5.5}
$$

Let us consider a set of ten commuting operators

$$
\hat{p}_\mu,\ \hat{W}^2,\ \hat{\mathbf{p}}\hat{\mathbf{S}},\ \hat{\mathbf{S}}^2 - \hat{\mathbf{B}}^2,\ \hat{\mathbf{S}}\hat{\mathbf{B}},\ \ \hat{S}^R_3,\ \hat{B}^R_3. \tag{5.6}
$$

This set consists of operators of momenta, the Lubanski-Pauli operator $\hat{W}^2$, the proportional to helicity operator $\hat{\mathbf{p}}\hat{\mathbf{J}} = \hat{\mathbf{p}}\hat{\mathbf{S}}$, and four operators, which are the functions of the right

generators. This four operators commute with the left rotations and translations and allow one to distinguish equivalent irreps in the decomposition of GRR. In the rest frame $\hat{\mathbf{p}}\hat{\mathbf{S}} = 0$, and the complete set of commuting operators can be obtained from (5.6) with the help of the replacement of $\hat{\mathbf{p}}\hat{\mathbf{S}}$ by $\hat{S}_3$.

Functions $f(x, \mathbf{z})$ on the group $M(3,1)$ are the functions of four real variables x^μ and four complex variables z_α, $\underline{z}_\alpha$ with the constraint $z_1\underline{z}_2 - z_2\underline{z}_1 = 1$.

The space of functions on the Poincaré group contains the subspace of analytic functions $f(x, z, \overset{*}{\underline{z}})$ of the elements of the Dirac z-spinor

$$Z_D = (z^\alpha, \overset{*}{\underline{z}}_{\dot\alpha}). \tag{5.7}$$

Charge conjugation means the transition to subspace of antianalytic functions (i.e. analytic functions of $\underline{z}^\alpha, \overset{*}{z}_{\dot\alpha}$).

According to (2.61), for the space inversion we have $Z \xrightarrow{P} (Z^{-1})^\dagger$ or

$$\begin{pmatrix} z^1 & \underline{z}^1 \\ z^2 & \underline{z}^2 \end{pmatrix} \xrightarrow{P} \begin{pmatrix} -\overset{*}{\underline{z}}_{\dot 1} & \overset{*}{z}_{\dot 1} \\ -\overset{*}{\underline{z}}_{\dot 2} & \overset{*}{z}_{\dot 2} \end{pmatrix}, \tag{5.8}$$

This transformation reverses the sign of the boost operators $\hat{B}_k$. It is easy to see that, in contrast to charge conjugation, space inversion conserves the analyticity (or antianalyticity) of functions of Z_D.

Similarly to three-dimensional case (see (4.5)), eigenfunctions of $\hat{S}_3^R$ and $\hat{B}_3^R$ differ only by a phase factor. Fixing eigenvalues of operators $\hat{S}_3^R$ and $\hat{B}_3^R$, one may pass to the space of functions of x^μ and elements of Majorana z-spinor

$$Z_M = (z^\alpha, \overset{*}{z}_{\dot\alpha}), \tag{5.9}$$

i.e. the space of functions of 8 real independent variables on the manifold

$$\mathbb{R}^4 \times \mathbb{C}^2, \qquad d\mu = d^4x d^2z_1 d^2z_2. \tag{5.10}$$

Thus in this space we have 8 commuting operators (2 Casimir operators, 4 operators distinguish states inside the irrep, 2 operators distinguish equivalent irreps). Notice that physical argumentation of necessity to make use at least 8 variables in order to describe spinning particles contains in [98]. The space reflection takes the functions of Z_M to the functions of $\underline{Z}_M = (\underline{z}^\alpha, \overset{*}{\underline{z}}_{\dot\alpha})$; as was mentioned above, $\underline{z}^\alpha$ and z^α have the same transformation rule. The charge conjugation leave the space of functions of Z_M invariant.

Below we will consider the massive case characterized by the symmetry with respect to space reflection and therefore the space of the analytic functions of Dirac z-spinor Z_D, unless otherwise stipulated. In this space the action of $M(3,1)$ is given by a formula

$$\begin{aligned} &T(g)f(x, z, \underline{z}) = f(g^{-1}x, g^{-1}z, g^{-1}\underline{z}), \\ &(g^{-1}x)^\mu = (\Lambda^{-1})^\mu{}_\nu x^\nu, \quad (g^{-1}z)^\alpha = U^\alpha{}_\beta z^\beta, \quad (g^{-1}\underline{z})_{\dot\alpha} = (\overset{*}{U}{}^{-1})_{\dot\alpha}{}^{\dot\beta} \overset{*}{\underline{z}}_{\dot\beta}. \end{aligned} \tag{5.11}$$

Spin projection operators have the form

$$\hat{S}_k = \frac{1}{2}(z\sigma_k\partial_z - \overset{*}{\underline{z}}\overset{*}{\sigma}_k\partial_{\overset{*}{\underline{z}}}), \quad \hat{B}_k = \frac{i}{2}(z\sigma_k\partial_z + \overset{*}{\underline{z}}\overset{*}{\sigma}_k\partial_{\overset{*}{\underline{z}}}). \tag{5.12}$$

It is known that one can compose the combinations $\hat{M}_k$, $\hat{\bar{M}}_k$:

$$\begin{aligned}
&\hat{M}_k = \frac{1}{2}(\hat{S}_k - i\hat{B}_k) = z\sigma_k\partial_z, \quad \hat{M}_+ = z^1\partial/\partial z^2, \quad \hat{M}_- = z^2\partial/\partial z^1,\\
&\hat{\bar{M}}_k = -\frac{1}{2}(\hat{S}_k + i\hat{B}_k) = \overset{*}{\underline{z}}\overset{*}{\sigma}_k\partial_{\overset{*}{\underline{z}}}, \quad \hat{\bar{M}}_+ = \overset{*}{\underline{z}}{}^{\dot{1}}\partial/\partial\overset{*}{\underline{z}}{}^{\dot{2}}, \quad \hat{\bar{M}}_- = \overset{*}{\underline{z}}{}^{\dot{2}}\partial/\partial\overset{*}{\underline{z}}{}^{\dot{1}},
\end{aligned} \tag{5.13}$$

such that $[\hat{M}_i, \hat{\bar{M}}_k] = 0$. For unitary representations of the Lorentz group $\hat{S}_k^\dagger = \hat{S}_k$, $\hat{B}_k^\dagger = \hat{B}_k$, and these operators obey the relation $\hat{M}_k^\dagger = \hat{\bar{M}}_k$ (for finite-dimensional nonunitary irreps $\hat{S}_k^\dagger = \hat{S}_k$, $\hat{B}_k^\dagger = -\hat{B}_k$ and $\hat{M}_k^\dagger = -\hat{\bar{M}}_k$). Introducing spin operators with spinor indices

$$\hat{M}_{\alpha\beta} = (\sigma_{\mu\nu})_{\alpha\beta}\hat{S}^{\mu\nu}, \quad \hat{\bar{M}}_{\dot{\alpha}\dot{\beta}} = (\bar{\sigma}_{\mu\nu})_{\dot{\alpha}\dot{\beta}}\hat{S}^{\mu\nu}, \tag{5.14}$$

where $\sigma_{\mu\nu}$ and $\bar{\sigma}_{\mu\nu}$ are defined in (B6), we obtain

$$\hat{S}^{\mu\nu} = -\frac{1}{2}\left((\sigma^{\mu\nu})^{\alpha\beta}\hat{M}_{\alpha\beta} + (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}\dot{\beta}}\hat{\bar{M}}_{\dot{\alpha}\dot{\beta}}\right), \tag{5.15}$$

$$\hat{M}_{\alpha\beta}\hat{M}^{\alpha\beta} = 2\hat{\mathbf{M}}^2, \quad \hat{\bar{M}}_{\dot{\alpha}\dot{\beta}}\hat{\bar{M}}^{\dot{\alpha}\dot{\beta}} = 2\hat{\bar{\mathbf{M}}}^2. \tag{5.16}$$

In the space of analytic functions of $z, \overset{*}{\underline{z}}$ we have:

$$\hat{M}_{\alpha\beta} = \frac{1}{2}(z_\alpha\partial_\beta + z_\beta\partial_\alpha), \quad \hat{\bar{M}}_{\dot{\alpha}\dot{\beta}} = \frac{1}{2}(\overset{*}{\underline{z}}_{\dot{\alpha}}\underline{\partial}_{\dot{\beta}} + \overset{*}{\underline{z}}_{\dot{\beta}}\underline{\partial}_{\dot{\alpha}}). \tag{5.17}$$

Taking into account that operators $\hat{M}_k$, $\hat{\bar{M}}_k$ are subjected to commutation relations of $su(2)$ algebra, we obtain spectra of the Casimir operators of the Lorentz subgroup:

$$\begin{aligned}
&\hat{\mathbf{S}}^2 - \hat{\mathbf{B}}^2 = 2(\hat{\mathbf{M}}^2 + \hat{\bar{\mathbf{M}}}^2) = 2j_1(j_1+1) + 2j_2(j_2+1) = -\frac{1}{2}(k^2 - \rho^2 - 4),\\
&\hat{\mathbf{S}}\hat{\mathbf{B}} = -i(\hat{\mathbf{M}}^2 - \hat{\bar{\mathbf{M}}}^2) = -i\left(j_1(j_1+1) - j_2(j_2+1)\right) = k\rho,\\
&\text{where} \quad \rho = -i(j_1 + j_2 + 1), \quad k = j_1 - j_2.
\end{aligned} \tag{5.18}$$

Thus irreps of the Lorentz group $SL(2,C)$ are labelled by the pair (j_1, j_2). It is convenient to label unitary irreps by $[k, \rho]$, where irreps $[k, \rho]$ and $[-k, -\rho]$ are equivalent [16,46].

Notice that the formulas (5.11)-(5.18) are also valid if, using substitution $\overset{*}{\underline{z}}_{\dot{\alpha}} \to \overset{*}{z}_{\dot{\alpha}}$, we consider the functions of elements of Majorana z-spinor Z_M instead of Z_D.

The formulas of reduction on the compact $SU(2)$-subgroup have the form

$$T_{(j_1,j_2)} = \sum_{j=|j_1-j_2|}^{j_1+j_2} T_j, \qquad T_{[k,\rho]} = \sum_{j=k}^{\infty} T_j \tag{5.19}$$

for finite-dimensional nonunitary irreps and infinite-dimensional unitary irreps of $SL(2,C)$ respectively [46]. Analogously with $2+1$ case, there are two types of the Poincaré group representations describing the same spin s. These types correspond to finite-dimensional

and infinite-dimensional unitary representations of the Lorentz group. In particular, one may choose: (i) $s = j_{\max} = j_1 + j_2$ for *nonunitary finite-dimensional* irreps (j_1, j_2); (ii) $s = j_{\min} = j_0 = |j_1 - j_2|$ for *unitary infinite-dimensional* irreps $[j_0, \rho]$, where $j_{\max}$ and $j_{\min}$ are maximal and minimal j in the decomposition (5.19) of an irrep of the Lorentz group over irreps T_j of compact $SU(2)$ subgroup. Below we will study only the case of finite-dimensional representations of the Lorentz group.

Consider monomial basis

$$(z^1)^a (z^2)^b (\overset{*}{\underline{z}}_1)^c (\overset{*}{\underline{z}}_2)^d$$

in the space of functions $\phi(z, \overset{*}{\underline{z}})$. The values $j_1 = (a+b)/2$ and $j_2 = (c+d)/2$ are conserved under the action of generators (5.13). Therefore the space of irrep (j_1, j_2) is the space of homogeneous analytic functions depending on two pairs of complex variables of power $(2j_1, 2j_2)$. We denote these functions as $\phi_{j_1 j_2}(z, \overset{*}{\underline{z}})$.

For finite-dimensional nonunitary irreps of $SL(2,C)$ a, b, c, d are integer nonnegative, therefore j_1, j_2 are integer or half-integer nonnegative numbers. One can write functions $f_s(x, z, \overset{*}{\underline{z}})$, which are polynomial of the power $2s = 2j_1 + 2j_2$ in $z, \overset{*}{\underline{z}}$, in the form

$$f_s(x, z, \overset{*}{\underline{z}}) = \sum_{j_1+j_2=s} \sum_{m_1, m_2} \psi^{m_1 m_2}_{j_1 j_2}(x) \varphi^{m_1 m_2}_{j_1 j_2}(z, \overset{*}{\underline{z}}), \tag{5.20}$$

where functions

$$\varphi^{m_1 m_2}_{j_1 j_2}(z, \overset{*}{\underline{z}}) = N^{\frac{1}{2}} (z^1)^{j_1+m_1} (z^2)^{j_1-m_1} (\overset{*}{\underline{z}}_{\dot{1}})^{j_2+m_2} (\overset{*}{\underline{z}}_{\dot{2}})^{j_2-m_2}, \tag{5.21}$$

$$N = (2s)![(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!]^{-1}, \tag{5.22}$$

form basis of the irrep of the Lorentz group. This basis corresponds to chiral representation (see Appendix B). On the other hand, one can write the decomposition of the same function in terms of symmetric multispinors $\psi^{\dot{\beta}_1 \ldots \dot{\beta}_{2j_2}}_{\alpha_1 \ldots \alpha_{2j_1}}(x) = \psi^{\dot{\beta}_{(1} \ldots \dot{\beta}_{2j_2)}}_{\alpha_{(1} \ldots \alpha_{2j_1)}}(x)$:

$$f_s(x, z, \overset{*}{\underline{z}}) = \sum_{j_1+j_2=s} f_{j_1 j_2}(x, z, \overset{*}{\underline{z}}), \quad f_{j_1 j_2}(x, z, \overset{*}{\underline{z}}) = \psi^{\dot{\beta}_1 \ldots \dot{\beta}_{2j_2}}_{\alpha_1 \ldots \alpha_{2j_1}}(x) z^{\alpha_1} \ldots z^{\alpha_{2j_1}} \overset{*}{\underline{z}}_{\dot{\beta}_1} \ldots \overset{*}{\underline{z}}_{\dot{\beta}_{2j_2}}. \tag{5.23}$$

Notice that similar generating functions summed over all s have been used in [69,70] to describe massless fields. Comparing decompositions (5.20) and (5.23), we obtain the relation

$$N^{\frac{1}{2}} \psi^{m_1 m_2}_{j_1 j_2}(x) = \psi^{\overbrace{\dot{1} \ldots \dot{1}}^{j_2+m_2} \overbrace{\dot{2} \ldots \dot{2}}^{j_2-m_2}}_{\underbrace{1 \ldots 1}_{j_1+m_1} \underbrace{2 \ldots 2}_{j_1-m_1}}(x). \tag{5.24}$$

Using invariant tensor $\sigma^\mu_{\dot{\alpha}\alpha}$ and spinors z^α, $\overset{*}{\underline{z}}_{\dot{\alpha}}$, $\partial_\alpha = \partial/\partial z^\alpha$, $\underline{\partial}^{\dot{\alpha}} = \partial/\partial \overset{*}{\underline{z}}_{\dot{\alpha}}$, it is possible to construct just four vectors:

$$\hat{V}^\mu_{12} = \frac{1}{2} \bar{\sigma}^{\mu \dot{\alpha} \alpha} \overset{*}{\underline{z}}_{\dot{\alpha}} \partial_\alpha, \quad \hat{V}^\mu_{21} = \frac{1}{2} \sigma^\mu{}_{\alpha\dot{\alpha}} z^\alpha \underline{\partial}^{\dot{\alpha}}, \tag{5.25}$$

$$\hat{V}^\mu_{11} = \frac{1}{2} \sigma^\mu{}_{\alpha\dot{\alpha}} z^\alpha \overset{*}{\underline{z}}^{\dot{\alpha}}, \quad \hat{V}^\mu_{22} = \frac{1}{2} \sigma^\mu{}_{\alpha\dot{\alpha}} \partial^\alpha \underline{\partial}^{\dot{\alpha}}. \tag{5.26}$$

These operators are not functions of generators of $M(3,1)$ and interlock irreps with different (j_1, j_2), however, as we will see below, play an impotent role in the theory of RWE.

B. Relativistic wave equations, invariant under proper Poincaré group

Let us fix eigenvalues of the Casimir operators of the Poincaré group and of the Lorentz subgroup:

$$\hat{p}^2 f(x,\mathbf{z}) = m^2 f(x,\mathbf{z}), \tag{5.27}$$
$$\hat{W}^2 f(x,\mathbf{z}) = -s(s+1)m^2 f(x,\mathbf{z}), \tag{5.28}$$
$$\hat{\mathbf{M}}^2 f(x,\mathbf{z}) = j_1(j_1+1) f(x,\mathbf{z}), \tag{5.29}$$
$$\hat{\bar{\mathbf{M}}}^2 f(x,\mathbf{z}) = j_2(j_2+1) f(x,\mathbf{z}), \tag{5.30}$$

Spectrum (5.28) of the operator $\hat{W}^2$ corresponds to the consideration of massive spin s particles and massless particles with discrete spin. (For tachyons and massless particles with continuous spin spectrum differ from (5.28), see [16,19].) As a consequence of two last equations (remind that we consider the space of analytic functions of $z, \underline{\overset{*}{z}}$) we obtain that eigenvalues of the belonging to the complete set (5.6) operators $\hat{S}^R_3$ and $\hat{B}^R_3$ are also fixed,

$$\hat{S}^R_3 f(x,z,\underline{\overset{*}{z}}) = -(j_1+j_2) f(x,z,\underline{\overset{*}{z}}), \quad i\hat{B}^R_3 f(x,z,\underline{\overset{*}{z}}) = (j_1-j_2) f(x,z,\underline{\overset{*}{z}}). \tag{5.31}$$

Equations (5.27)-(5.30) define reducible representation of the proper Poincaré group $M(3,1)$. This representation splits into two representations labelled by the sign of p_0, which are irreducible for $m \neq 0$.

Nonequivalent representations are distinguished by eigenvalues of the Casimir operators $\hat{p}^2$, $\hat{W}^2$ and by the sign of p_0 (see also [16,19,20]). The case of zero eigenvalues of the operators $\hat{p}^2$ and $\hat{W}^2$ is an exception. This case corresponds to massless particles with discrete spin, and nonequivalent irreps are labelled by the helicity and by the sign of p_0. On the other hand, one can introduce a chirality as $\lambda = j_1 - j_2$ (or as the difference of numbers of dotted and undotted indices). The explicit form of the chirality operator in the space of analytic functions of $z, \underline{\overset{*}{z}}$ is given by the formula (see (B4))

$$\hat{\Gamma}^5 = \tfrac{1}{2}\left(z^\alpha \partial_\alpha - \underline{\overset{*}{z}}_{\dot\alpha} \underline{\partial}^{\dot\alpha}\right). \tag{5.32}$$

In the massless case helicity is equal to chirality up to sign [16]. In the massive case irreps of the proper Poincaré group, which are labelled by the same $m, s, \operatorname{sign} p_0$ but by different chiralities, are equivalent. Thus, for fixed mass m and spin $s = j_1 + j_2$ the system (5.27)-(5.30) has $2s+1$ solutions differed by $\lambda = j_1 - j_2$.

Using (5.15), we rewrite the Lubanski-Pauli vector (5.4) and the Casimir operator $\hat{W}^2$ in the form

$$\hat{W}^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\hat{p}_\nu \hat{S}_{\rho\sigma} = \frac{1}{2} i\hat{p}_\nu \left((\sigma^{\mu\nu})_{\alpha\beta}\hat{M}^{\alpha\beta} - (\bar{\sigma}^{\mu\nu})_{\dot\alpha\dot\beta}\hat{\bar{M}}^{\dot\alpha\dot\beta}\right), \tag{5.33}$$
$$\hat{W}^2 = -\hat{p}^2(\hat{\mathbf{M}}^2 + \hat{\bar{\mathbf{M}}}^2) - \frac{1}{2}\hat{p}_\mu\hat{p}_\nu(\sigma^{\mu\rho})_{\alpha\beta}(\bar{\sigma}_\rho{}^\nu)_{\dot\alpha\dot\beta}\hat{M}^{\alpha\beta}\hat{\bar{M}}^{\dot\alpha\dot\beta}. \tag{5.34}$$

Taking into account the explicit form of spin operators (5.17) and symmetry of $(\sigma^{\mu\nu})_{\alpha\beta}$ and $(\bar{\sigma}^{\mu\nu})_{\dot\alpha\dot\beta}$ with respect to permutation of spinor indices, we rewrite the last relation as

$$\hat{W}^2 = -\hat{p}^2(\hat{\mathbf{M}}^2 + \hat{\bar{\mathbf{M}}}^2) - 2\hat{p}_\mu\hat{p}_\nu(\sigma^{\mu\rho})_{\alpha\beta}(\bar{\sigma}_\rho{}^\nu)_{\dot\alpha\dot\beta} z^\alpha \partial^\beta \underline{\overset{*}{z}}^{\dot\alpha} \underline{\partial}^{\dot\beta}.$$

Finally, using the identity

$$4(\sigma^{\mu\rho})_{\alpha\beta}(\bar{\sigma}_{\rho}{}^{\nu})_{\dot{\alpha}\dot{\beta}} = -\eta^{\mu\nu}\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} + \sigma_{\alpha\dot{\alpha}}{}^{\mu}\sigma_{\beta\dot{\beta}}{}^{\nu} + \sigma_{\alpha\dot{\alpha}}{}^{\nu}\sigma_{\beta\dot{\beta}}{}^{\mu}$$

and the condition of mass irreducibility (5.27), we obtain

$$\hat{W}^2 = -m^2(j_1+j_2)(j_1+j_2+1) + 4\hat{p}_\mu\hat{V}_{11}^{\mu}\hat{p}_\nu\hat{V}_{22}^{\nu}, \qquad (5.35)$$

where operators $\hat{V}_{11}^{\mu}$ and $\hat{V}_{22}^{\mu}$ are defined in (5.26). Therefore for $s = j_1 + j_2$ the necessary and sufficient condition of spin irreducibility is

$$\hat{p}_\mu\hat{V}_{11}^{\mu}\hat{p}_\nu\hat{V}_{22}^{\nu}f(x,z,\overset{*}{\underline{z}}) = 0. \qquad (5.36)$$

For the representations $(s\,0)$ and $(0\,s)$ we have $\hat{V}_{22}^{\mu}f(x,z,\overset{*}{\underline{z}}) = 0$ and the condition (5.36) is fulfilled identically. In general case, observing that in momentum representation an action of operator $\hat{V}_{11}^{\mu}\hat{p}_\mu$ reduces to multiplication by the number $p_\mu\sigma^{\mu}{}_{\alpha\dot{\alpha}}z^\alpha z^{\dot{\alpha}}$, we come to the *alternative conditions*:

$$\hat{p}_\mu\hat{V}_{11}^{\mu} = 0, \qquad (5.37)$$

$$\hat{p}_\nu\hat{V}_{22}^{\nu}f(x,z,\overset{*}{\underline{z}}) = 0. \qquad (5.38)$$

The first condition connects the components of momentum p_μ and complex spin variables $q_\mu = \sigma_{\mu\alpha\dot{\alpha}}z^\alpha\overset{*}{\underline{z}}{}^{\dot{\alpha}}/2$, $q_\mu q^\mu = 0$. Thus we have the space of functions of two 4-vectors p_μ, q_μ, which are subject to the invariant constraints

$$p^2 = m^2, \quad p_\mu q^\mu = 0, \quad q^2 = 0. \qquad (5.39)$$

According to (5.39), in the rest frame we get $z^1\overset{*}{\underline{z}}{}^1 + z^2\overset{*}{\underline{z}}{}^2 = 0$. Similar approach to the constructing of wave functions describing the elementary particles was suggested by E.Wigner in [35], where discussion was restricted to particles of integer spin and real q_μ with constraints $p^2 = m^2$, $p_\mu q^\mu = 0$, $q^2 = -1$. Different generalizations of the approach [35] were considered later in [36–40].

The second condition (5.38) does not affect spin variables and can be written in terms of $\psi(x)$. Really, for fixed j_1, j_2, using the decomposition (5.23) of $f(x,z,\overset{*}{\underline{z}})$ in terms of multispinors and also the relation $\partial_\alpha\psi_{\alpha_1\alpha_2\ldots\alpha_k}z^{\alpha_{(1}}z^{\alpha_2}\ldots z^{\alpha_{k)}} = \delta_\alpha{}^{\alpha_1}\psi_{\alpha_1\alpha_2\ldots\alpha_k}z^{\alpha_{(2}}\ldots z^{\alpha_{k)}}$, one can rewrite the system

$$(\hat{p}^2 - m^2)f_{j_1j_2}(x,z) = 0, \quad \hat{p}_\mu\sigma^{\mu}{}_{\alpha\dot{\alpha}}\partial^\alpha\underline{\partial}^{\dot{\alpha}}f_{j_1j_2}(x,z) = 0 \qquad (5.40)$$

in the form

$$(\hat{p}^2 - m^2)\psi_{\alpha_1\ldots\alpha_k\dot{\alpha}_1\ldots\dot{\alpha}_l}(x) = 0, \qquad (5.41)$$

$$\partial^{\dot{\alpha}\alpha}\psi_{\alpha\alpha_1\ldots\alpha_{k-1}\dot{\alpha}\dot{\alpha}_1\ldots\dot{\alpha}_{l-1}}(x) = 0, \qquad (5.42)$$

where $\partial^{\dot{\alpha}\alpha} = \partial_\mu\bar{\sigma}^{\mu\dot{\alpha}\alpha}$, $k = 2j_1$, $l = 2j_2$. These equations describe a particle with unique mass m and spin $s = j_1 + j_2$. Subsidiary condition (5.42) is necessary to exclude components corresponding to other possible spins s, $|j_1 - j_2| \le s < j_1 + j_2$, see (5.19).

On the other hand, in order to describe spin s one may use representations $(j_1\, j_2)$, $j_1 + j_2 \neq s$. In this case, according to (5.35), the condition (5.42) should be changed by new subsidiary condition

$$\partial_{\beta\dot\beta}\partial^{\dot\alpha\alpha}\psi_{\alpha\alpha_1\ldots\alpha_{k-1}\dot\alpha\dot\alpha_1\ldots\dot\alpha_{l-1}}(x) = -m^2[(j_1+j_2)(j_1+j_2+1) - s(s+1)]\psi_{\beta\alpha_1\ldots\alpha_{k-1}\dot\beta\dot\alpha_1\ldots\dot\alpha_{l-1}}(x). \tag{5.43}$$

Notice that an approach using this general subsidiary conditions was not considered earlier.

Passing on to vector indices, one can see that for integer spins and irreps $(\frac{s}{2}\,\frac{s}{2})$ equations (5.41), (5.42) take the form

$$(\hat p^2 - m^2)\Phi_{\mu_1\mu_2\ldots\mu_s}(x) = 0, \quad \partial^\mu\Phi_{\mu\mu_2\ldots\mu_s}(x) = 0, \quad \Phi^\mu{}_{\mu\ldots\mu_s}(x) = 0, \tag{5.44}$$

where

$$\Phi_{\mu_1\mu_2\ldots\mu_s}(x) = (-1)^s 2^{-s}\bar\sigma_{\mu_1}{}^{\dot\alpha_1\alpha_1}\ldots\bar\sigma_{\mu_s}{}^{\dot\alpha_s\alpha_s}\psi_{\alpha_1\ldots\alpha_s\dot\alpha_1\ldots\dot\alpha_s}(x).$$

Just equations (5.44) known also as massive tensor field equations or Fierz–Pauli equations are used most often to describe integer spins.

For half-integer spins and irreps $(\frac{2s\pm1}{4}\;\frac{2s\mp1}{4})$ after passage to vector indices subsidiary conditions (5.42) take the form

$$\begin{aligned}
&\partial^\mu\Psi_{\mu\mu_2\ldots\mu_n\alpha}(x) = 0,\ \bar\sigma^{\mu\dot\alpha\alpha}\Psi_{\mu\mu_2\ldots\mu_n\alpha}(x) = 0,\ \Psi^\mu{}_{\mu\mu_2\ldots\mu_n\alpha}(x) = 0,\\
&\partial^\mu\Psi_{\mu\mu_2\ldots\mu_n\dot\alpha}(x) = 0,\ \sigma^\mu{}_{\alpha\dot\alpha}\Psi_{\mu\mu_2\ldots\mu_n}{}^{\dot\alpha}(x) = 0,\ \Psi^\mu{}_{\mu\mu_2\ldots\mu_n\dot\alpha}(x) = 0,
\end{aligned} \tag{5.45}$$

where $n = (2s-1)/2$.

C. Relativistic wave equations, invariant under improper Poincaré group

Improper Poincaré group includes continuous transformations of the proper group and space reflection operator (parity operator) $\hat I_P$. According to (2.61),(5.8) this operator obeys the condition $\hat I_P^2 = \hat 1$ and commutation relations

$$[\hat I_P, \hat p_0] = [\hat I_P, \hat p^2] = [\hat I_P, \hat W^2] = [\hat I_P, \hat S_k] = [\hat I_P, \hat S_k^R] = 0, \tag{5.46}$$

$$[\hat I_P, \hat p_k]_+ = [\hat I_P, \hat B_k]_+ = [\hat I_P, \hat B_k^R]_+ = 0. \tag{5.47}$$

States with definite total parity are defined as eigenfunctions of operator $\hat I_P$:

$$\hat I_P f(x, \mathbf{z}) = \pm f(x, \mathbf{z}). \tag{5.48}$$

For $m > 0$ irreps of the improper Poincaré group are labelled by an orbit defining the mass m and the sign of p_0, and by irrep of the little group $O(3)$ defining spin s and intrinsic parity [16,19,99]. In the rest frame the intrinsic parity coincides with the total.

The Casimir operators of the Lorentz group do not commute with parity operator, $[\hat I_P, \hat{\mathbf{M}}^2] = \hat{\tilde{\mathbf{M}}}^2$, $[\hat I_P, \hat{\tilde{\mathbf{M}}}^2] = \hat{\mathbf{M}}^2$, and parity transformation combines two labelled by Lorentz indices (j_1, j_2) and (j_2, j_1) (by chiralities $\pm\lambda$) equivalent irreps of the proper Poincaré group into one representation of the improper group. The latter representation is reductible and splits into two irreps differed by intrinsic parity $\eta = \pm 1$. Thus we can't make use the

operators $\hat{\mathbf{M}}^2$, $\hat{\underline{\mathbf{M}}}^2$ to select invariant subspaces, and instead of the set of eight commuting operators

$$\hat{p}_\mu, \hat{W}^2,\ \hat{\mathbf{p}}\hat{\mathbf{S}},\ \hat{\mathbf{M}}^2,\ \hat{\underline{\mathbf{M}}}^2 \tag{5.49}$$

used above in order to construct the system (5.27)-(5.30) we should consider an another set. Notice that parity operator $\hat{I}_P$ can't be used directly for identification of invariant subspaces since according to (5.47) it does not commute with translations and boosts.

The simplest possibility is to consider a system

$$\hat{p}^2 f(x,z,\overset{*}{\underline{z}}) = m^2 f(x,z,\overset{*}{\underline{z}}), \tag{5.50}$$

$$\hat{W}^2 f(x,z,\overset{*}{\underline{z}}) = -s(s+1)m^2 f(x,z,\overset{*}{\underline{z}}), \tag{5.51}$$

$$\hat{S}_3^R f(x,z,\overset{*}{\underline{z}}) = -s f(x,z,\overset{*}{\underline{z}}). \tag{5.52}$$

The last equation fixes the power $2s = 2(j_1+j_2)$ of the polynomial in $z, \overset{*}{\underline{z}}$, see (5.31). The first two equations are the conditions of mass and spin irreducibility. Therefore the system describes fixed mass and spin, but the Poincaré group representation defined by this system is reducible. This representation decomposes into $2(2s+1)$ irreps differed by the chirality $\lambda = -s, \ldots, s$ and sign of p_0.

Supplementing the system (5.50)-(5.52) by the equation

$$i\hat{B}_3^R f(x,z,\overset{*}{\underline{z}}) = \pm(j_1-j_2) f(x,z,\overset{*}{\underline{z}}), \tag{5.53}$$

which change the sign under space reflection, it is possible to extract components corresponding to the representation $(j_1,j_2)\oplus(j_2,j_1)$. If we consider only the components labelled by (j_1,j_2) and (j_2,j_1), then for $j_1 \neq j_2$ mass and spin irreducibility conditions (5.50),(5.51) leave $4(2s+1)$ independent components corresponding to the direct sum of four improper Poincaré group irreps differed by signs of energy p_0 and intrinsic parity η. But states with definite intrinsic parity arise in such an approach only as linear combinations of the solutions of *two* systems (5.50)-(5.53) with different sign in the last equation (i.e. solutions with fixed chirality).

Let us investigate the possibility to construct the system of equations, which remains invariant under space reflection and has solutions with definite intrinsic parity. For this purpose it is necessary to consider equations, which combine labelled by different chiralities $\lambda = j_1 - j_2$ equivalent irreps of the proper Poincaré group. In the other words, *it is necessary to consider supplementary operators, which define some extension of the Lorentz group.* These operators, replacing $\hat{\mathbf{M}}^2$ and $\hat{\underline{\mathbf{M}}}^2$ in the set (5.49), must commute with all the left generators of the proper Poincaré group and with parity operator $\hat{I}_P$. *We suppose that one of these commuting operators is linear in $\hat{p}$.*

A general form of the invariant equations linear in $\hat{p}$ is

$$\hat{p}_\mu \hat{V}^\mu f(x,\mathbf{z}) = \varkappa f(x,\mathbf{z}), \tag{5.54}$$

where $\hat{V}^\mu$ is a transforming as four-vector function of $\mathbf{z}$ and $\partial/\partial\mathbf{z}$.

The introduced above vector operators V^μ_{ik} (5.25), (5.26) interlock irreps with different (j_1,j_2). Operators $\hat{V}^\mu_{12}$, $\hat{V}^\mu_{21}$ conserve j_1+j_2, and operators $\hat{V}^\mu_{11}$, $\hat{V}^\mu_{22}$ conserve j_1-j_2. Any of four connecting two scalar functions relations

$$\hat{p}_\mu \hat{V}^\mu_{12} f_{j_1,j_2}(x,z,\overset{*}{\underline{z}}) = \varkappa_{12} f_{j_1-\frac{1}{2},j_2+\frac{1}{2}}(x,z,\overset{*}{\underline{z}}),\ \hat{p}_\mu \hat{V}^\mu_{21} f_{j_1,j_2}(x,z,\overset{*}{\underline{z}}) = \varkappa_{21} f_{j_1+\frac{1}{2},j_2-\frac{1}{2}}(x,z,\overset{*}{\underline{z}}), \quad (5.55)$$

$$\hat{p}_\mu \hat{V}^\mu_{11} f_{j_1,j_2}(x,z,\overset{*}{\underline{z}}) = \varkappa_{11} f_{j_1+\frac{1}{2},j_2+\frac{1}{2}}(x,z,\overset{*}{\underline{z}}),\ \hat{p}_\mu \hat{V}^\mu_{22} f_{j_1,j_2}(x,z,\overset{*}{\underline{z}}) = \varkappa_{22} f_{j_1-\frac{1}{2},j_2-\frac{1}{2}}(x,z,\overset{*}{\underline{z}}), \quad (5.56)$$

one may consider as a RWE. Thus the operator $\hat{V}^\mu$ in (5.54) is a linear combination of $\hat{V}^\mu_{ik}$.

Let us consider finite-component equations invariant with respect to space reflection. This means:

1. The operator $\hat{p}_\mu \hat{V}^\mu$ is invariant under space reflection.

2. The equation has solutions $f(x,z,\overset{*}{\underline{z}}) = \sum \psi_n(x)\phi_n(z,\overset{*}{\underline{z}})$, where functions $\phi_n(z)$ carry a representation containing finite number of irreps (j_1,j_2).

It is easy to see that at $\varkappa_{11} \neq 0$ operator $\hat{V}^\mu_{11}$ can't be contained in $\hat{V}^\mu$. In this case at $\varkappa_{22} \neq 0$ one can separate from the system of equations for functions $f_{j_1,j_2}(x,z,\overset{*}{\underline{z}})$, $f(x,z,\overset{*}{\underline{z}}) = \sum f_{j_1,j_2}(x,z,\overset{*}{\underline{z}})$ the independent equation for the characterized by maximal j_1+j_2 function, which does not contain $\hat{V}^\mu_{22}$. (Besides, it is not necessary to use operators $\hat{V}^\mu_{11}$ and $\hat{V}^\mu_{22}$ since these operators leave $j_1 - j_2$ invariable and can't connect irreps with different λ.)

Relating to operators $\hat{V}^\mu_{12}$ and $\hat{V}^\mu_{21}$, one can see that only the combination $\hat{p}_\mu \hat{\Gamma}^\mu$,

$$\hat{\Gamma}^\mu = \hat{V}^\mu_{12} + \hat{V}^\mu_{21} = \frac{1}{2}\left(\bar{\sigma}^{\mu\dot{\alpha}\alpha}\overset{*}{\underline{z}}_{\dot{\alpha}}\partial_\alpha + \sigma^\mu{}_{\alpha\dot{\alpha}} z^\alpha \underline{\partial}^{\dot{\alpha}}\right), \quad (5.57)$$

is invariant under space reflections. Operators $\hat{\Gamma}^\mu$ connect representation $(j_1\, j_2)$ with $(j_1+1\ j_2-1)$ and $(j_1-1\ j_2+1)$ and conserve j_1+j_2. These operators obey the commutation relations

$$[\hat{S}^{\lambda\mu}, \hat{\Gamma}^\nu] = i(\eta^{\mu\nu}\hat{\Gamma}^\lambda - \eta^{\lambda\nu}\hat{\Gamma}^\mu), \quad (5.58)$$

$$[\hat{\Gamma}^\mu, \hat{\Gamma}^\nu] = -i\hat{S}^{\mu\nu}, \quad (5.59)$$

which coincide with the commutation relations of matrices $\gamma^\mu/2$. The explicit calculation shows that $\hat{\Gamma}_\mu\hat{\Gamma}^\mu$ depends on irrep of the Lorentz subgroup,

$$\hat{\Gamma}_\mu\hat{\Gamma}^\mu = 2j_1 + 2j_2 + 4j_1 j_2. \quad (5.60)$$

Supplementing generators of the Lorentz group by four operators

$$\hat{S}^{4\mu} = \hat{\Gamma}^\mu, \quad \hat{S}^{ab} = -\hat{S}^{ba}, \quad (5.61)$$

we obtain

$$[\hat{S}^{ab}, \hat{S}^{cd}] = i(\eta^{bc}\hat{S}^{ad} - \eta^{ac}\hat{S}^{bd} - \eta^{bd}\hat{S}^{ac} + \eta^{ad}\hat{S}^{bc}), \quad \eta^{44} = \eta^{00} = 1. \quad (5.62)$$

Thus operators $\hat{S}^{ab}$, $a,b = 0,1,2,3,4$, obey the commutation relations of the generators of the 3+2 de Sitter group $SO(3,2) \sim Sp(4,R)$. This group has two fundamental irreps, namely four-dimensional spinor irrep $T_{[10]}$ (by matrices $Sp(4,R)$) and five-dimensional vector irrep $T_{[01]}$ (by matrices $SO(3,2)$).

Using (5.5), (5.18) and (5.60), we obtain for the second order Casimir operator of the group $Sp(4,R)$

$$\hat{S}_{ab}\hat{S}^{ab} f(x,z,\overset{*}{\underline{z}}) = 4S(S+2) f(x,z,\overset{*}{\underline{z}}), \quad S = j_1 + j_2.$$

Thus we deal with symmetric representations of $Sp(4,R)$, which we denote as $T_{[2S\,0]}$ (see Appendix). These irreps can be obtained as a symmetric term in the decomposition of the direct product $(\otimes T_{[1\,0]})^{2S}$. Irreps $T_{[2S\,0]}$ characterized by dimensionality $(2S+3)!/(6(2S)!)$ combines all finite-dimensional irreps of the Lorentz group with $j_1+j_2=S$.

However, it is obvious that the equation

$$\hat{p}_\mu \hat{\Gamma}^\mu f(x,z,\underline{\overset{*}{z}}) = \varkappa f(x,z,\underline{\overset{*}{z}}) \tag{5.63}$$

by itself does not fix spin s and mass m, defined by (5.27) and (5.28), or the power j_1+j_2 of the $f(x,z,\underline{\overset{*}{z}})$ in $z,\underline{\overset{*}{z}}$. In the rest frame it is easy to see that even for fixed $S=j_1+j_2$ this equation fix only product $ms=\varkappa$, $s\le S$.

Let us consider the set of eight commuting operators

$$\hat{p}_\mu,\ \hat{W}^2,\ \hat{\mathbf{p}}\hat{\mathbf{S}}\ (\text{or } \hat{S}_3 \text{ in the rest frame}),\ \hat{p}_\mu\hat{\Gamma}^\mu,\ \hat{S}_{ab}\hat{S}^{ab} \tag{5.64}$$

acting in the space of functions of eight variables $x^\mu, z^\alpha, \underline{\overset{*}{z}}_{\dot\alpha}$. In compare with the set (5.49) we have replaced two right operators $\mathbf{M}^2$, $\hat{\tilde{\mathbf{M}}}^2$ by invariant under parity transformation operators $\hat{p}_\mu\hat{\Gamma}^\mu$, $\hat{S}_{ab}\hat{S}^{ab}$. Notice that instead of $\hat{S}_{ab}\hat{S}^{ab}$ one can use operator $\hat{S}^R_3$ with eigenvalues equal to the minus power of polynomial in $z,\underline{\overset{*}{z}}$, see (5.52).

Invariant subspaces are labelled by eigenvalues of operators

$$\hat{p}^2 f(x,z,\underline{\overset{*}{z}}) = m^2 f(x,z,\underline{\overset{*}{z}}), \tag{5.65}$$

$$\hat{W}^2 f(x,z,\underline{\overset{*}{z}}) = -m^2 s(s+1) f(x,z,\underline{\overset{*}{z}}), \tag{5.66}$$

$$\hat{p}_\mu\hat{\Gamma}^\mu f(x,z,\underline{\overset{*}{z}}) = m\tilde{s} f(x,z,\underline{\overset{*}{z}}), \tag{5.67}$$

$$\hat{S}_{ab}\hat{S}^{ab} f(x,z,\underline{\overset{*}{z}}) = 4S(S+2) f(x,z,\underline{\overset{*}{z}}). \tag{5.68}$$

Unlike equations (5.29),(5.30), which fix j_1 and j_2 separately, the last equation of the system fixes irrep $T_{[2S\,0]}$ of the 3+2 de Sitter group and therefore the power $2S=2j_1+2j_2$ of the polynomial in $z,\underline{\overset{*}{z}}$. Irreps of the Poincaré group characterized by spin $s\le S$ can be realized in the space of these polynomials.

In the rest frame

$$\begin{aligned}
&\hat{p}_0^2 f(x,z,\underline{\overset{*}{z}}) = m^2 f(x,z,\underline{\overset{*}{z}}),\\
&\hat{p}_0\hat{\Gamma}^0 f(x,z,\underline{\overset{*}{z}}) = m\tilde{s} f(x,z,\underline{\overset{*}{z}}), \quad \hat{\Gamma}^0 = \frac{1}{2}(\sigma^{0\dot\alpha\alpha}\underline{\overset{*}{z}}_{\dot\alpha}\partial_\alpha + \sigma^0{}_{\alpha\dot\alpha} z^\alpha \underline{\overset{*}{\partial}}{}^{\dot\alpha}).
\end{aligned} \tag{5.69}$$

According to the first equation, $p_0=\pm m$ and correspondingly $\tilde{s}$ is a product of eigenvalue of operator $\hat{\Gamma}^0$ and $\operatorname{sign} p_0$. For $p_0=m$, any characterized by $n_1-n_2=2s$ function is the solution of equation (5.69), where n_1 is the power of homogeneity in the variables $(z^1+\underline{\overset{*}{z}}_{\dot 1})$, $(z^2+\underline{\overset{*}{z}}_{\dot 2})$, and n_2 is the power of homogeneity in the variables $(z^1-\underline{\overset{*}{z}}_{\dot 1})$, $(z^2-\underline{\overset{*}{z}}_{\dot 2})$. Therefore, for $p_0=-m$ any characterized by $n_1-n_2=-2s$ function is the solution of equation (5.69).

Let us show that the relation

$$|\tilde{s}| \le s \le S \tag{5.70}$$

takes place. Variables z^α and $z_{\dot\alpha}$ have the same transformation rule under space rotations. Thus, the pairs $(z^1+\underline{\overset{*}{z}}_{\dot 1})$, $(z^2+\underline{\overset{*}{z}}_{\dot 2})$ and $(z^1-\underline{\overset{*}{z}}_{\dot 1})$, $(z^2-\underline{\overset{*}{z}}_{\dot 2})$ are spinors of rotation group, but are

characterized by opposite parity. The polynomials of power $2j'$ in the first pair of variables or $2j''$ in the second pair of variables transform under $T_{j'}$ or $T_{j''}$ of the rotation group. At fixed j' and j'' the relation $\tilde{s} = (j' - j'')\,\mathrm{sign}\, p_0$ takes place, and the space of polynomials of the power $2S = 2j' + 2j''$ corresponds to direct product of the representations $T_{j'}$ and $T_{j''}$. This direct product decomposes into sum of irreps, labelled by $s = |j' - j''|, \dots, j' + j''$, and therefore spin s runs from $|\tilde{s}|$ up to S.

In particular, for $|\tilde{s}| = S$ the spin irreducibility condition (5.66) is a consequence of other equations of the system, and spin is equal to one half of the polynomial power. Below we restrict our consideration by this case, which allows to describe spin s by means of the irrep of the 3+2 de Sitter group with minimal possible dimensionality. Correspondingly, for $\tilde{s} = S$ we will consider the system

$$\hat{p}^2 f(x, \mathbf{z}) = m^2 f(x, \mathbf{z}), \tag{5.71}$$

$$\hat{p}_\mu \hat{\Gamma}^\mu f(x, \mathbf{z}) = m s f(x, \mathbf{z}), \tag{5.72}$$

$$\hat{S}_{ab} \hat{S}^{ab} f(x, \mathbf{z}) = 4s(s+2) f(x, \mathbf{z}). \tag{5.73}$$

In the rest frame the general solution in the set of polynomial of the power $2s$ in $z, \overset{*}{\underline{z}}$ has the form

$$f_{m,s}(x, \mathbf{z}) = \sum_{s_3=-s}^{s} C_{s_3} e^{imx^0} (z^1 + \overset{*}{\underline{z}}_{\dot{1}})^{s+s_3} (z^2 + \overset{*}{\underline{z}}_{\dot{2}})^{s-s_3} + C'_{s_3} e^{-imx^0} (z^1 - \overset{*}{\underline{z}}_{\dot{1}})^{s+s_3} (z^2 - \overset{*}{\underline{z}}_{\dot{2}})^{s-s_3}, \tag{5.74}$$

where s_3 is the spin projection,

$$\hat{S}_3 f(x, \mathbf{z}) = s_3 f(x, \mathbf{z}), \quad \hat{S}_3 = \frac{1}{2}(z^1 \partial_1 + \overset{*}{\underline{z}}_{\dot{1}} \underline{\partial}^{\dot{1}} - z^2 \partial_2 - \overset{*}{\underline{z}}_{\dot{2}} \underline{\partial}^{\dot{2}}). \tag{5.75}$$

Thus for unique mass m and spin s there are $2s+1$ independent positive-frequency solutions and $2s+1$ independent negative-frequency solutions belonging to two irreps of improper Poincaré group. In the case $\tilde{s} = -S$, which corresponds to the change of sign in the equation (5.72), general solution is obtained from (5.74) by the substitution $(z^\alpha + \overset{*}{\underline{z}}_{\dot{\alpha}}) \leftrightarrow (z^\alpha - \overset{*}{\underline{z}}_{\dot{\alpha}})$. It follows from (5.74) that for half-integer spins the sign of $\tilde{s}$ is the product of $\mathrm{sign}\, p_0$ and intrinsic parity. [6]

Only corresponding to spin 1/2 four-dimensional irrep of the 3+2 de Sitter group remains irreducible under the reduction on the improper Lorentz group. For spin one 10-dimensional irrep splits into 6+4 (antisymmetric tensor and four-vector), for spin 3/2 20-dimensional irrep splits into 8+12, and so on.

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[6] According to (5.74), in the rest frame for half-integer spin positive-frequency and negative-frequency states are characterized by opposite parity. One can show (see [100–102]) that for fixed mass m and representation $(\frac{1}{2}\,0) \oplus (0\,\frac{1}{2})$ of the Lorentz group this condition is sufficient to derive the Dirac equation.

Consider plain wave solutions corresponding to a moving along x^3 particle. They can be obtained from the solutions in the rest frame (5.74) by means of the Lorentz transformation

$$P = UP_0U^\dagger, \quad \text{where } P_0 = \pm\,\mathrm{diag}\{m, m\}, \quad U = \mathrm{diag}\{e^{-a}, e^a\} \in SL(2, C),$$

where the sign corresponds to the sign of p_0,

$$p_\mu = k_\mu \mathrm{sign}\ p_0, \quad k_0 = m\cosh 2a, \quad k_3 = m\sinh 2a, \quad e^{\pm a} = \sqrt{(k_0 \pm k_3)/m}. \tag{5.76}$$

Thus it follows that

$$\begin{aligned} f'_{m,s,s_3}(x, \mathbf{z}) = C_1 e^{ik_0x^0+k_3x^3}(z^1e^a + \underline{\overset{*}{z}}_{\dot 1}e^{-a})^{s+s_3}(z^2e^{-a} + \underline{\overset{*}{z}}_{\dot 2}e^{a})^{s-s_3} + \\ C_2 e^{-ik_0x^0-k_3x^3}(z^1e^a - \underline{\overset{*}{z}}_{\dot 1}e^{-a})^{s+s_3}(z^2e^{a} - \underline{\overset{*}{z}}_{\dot 2}e^{-a})^{s-s_3}. \end{aligned} \tag{5.77}$$

In the ultrarelativistic case for positive a it is convenient to rewrite (5.77) in the form

$$\begin{aligned} & f_{m,s,s_3}(x, \mathbf{z}) = \left(\tfrac{k_0+k_3}{m}\right)^s \times \\ & \left(\left(C_1 e^{ik_0x^0+k_0x^3} + C_2(-1)^{s-s_3}e^{-ik_0x^0-k_0x^3}\right)(z^1)^{s+s_3}(\underline{\overset{*}{z}}_{\dot 2})^{s-s_3} + O\left(\tfrac{k_0-k_3}{k_0+k_3}\right)^{\frac{1}{2}}\right) \end{aligned} \tag{5.78}$$

The main term in (5.78) corresponds to functions carrying irrep $(\frac{s+\lambda}{2}\ \frac{s-\lambda}{2})$, $\lambda = s_3$, of the Lorentz group. The contribution of other irreps $(\frac{s+\lambda'}{2}\ \frac{s-\lambda'}{2})$ are damped by a factor $(\frac{k_0-k_3}{k_0+k_3})^{|\lambda-\lambda'|}$. Passing to the limit $a \to +\infty$ (or $m \to 0$), we obtain the states with certain chirality $\lambda = j_1 - j_2 = s_3$ (for $a \to -\infty$ with chirality $\lambda = j_1 - j_2 = -s_3$ respectively). In particular, in the limit the states characterized by helicity $s_3 = \pm s$ correspond to the representation $(s\,0) \oplus (0\,s)$ of the Lorentz group.

Taking into account that operators $\hat V^\mu_{21}$ $(\hat V^\mu_{12})$ lower (raise) chirality λ by 1 and the decomposition

$$f_s(x, z, \underline{\overset{*}{z}}) = \sum_{\lambda=-s}^{s} f_{j_1j_2}(x, z, \underline{\overset{*}{z}}), \quad \text{where } s = j_1 + j_2, \quad \lambda = j_1 - j_2, \tag{5.79}$$

one can write equation (5.72) in chiral representation in the form

$$\begin{pmatrix} \hat p_\mu \hat V^\mu_{21} f_{s-\frac12,\frac12} \\ \hat p_\mu \hat V^\mu_{12} f_{s,0} + \hat p_\mu \hat V^\mu_{21} f_{s-1,1} \\ \dots \\ \hat p_\mu \hat V^\mu_{12} f_{\frac12,s-\frac12} \end{pmatrix} = ms \begin{pmatrix} f_{s,0} \\ f_{s-\frac12,\frac12} \\ \dots \\ f_{0,s} \end{pmatrix}. \tag{5.80}$$

For $m \neq 0$ this equation binds $1 + [s]$ irreps of the improper Lorentz group and allows one to express components corresponding to irrep $(s\ 0)$ in terms of components corresponding to irrep $(s - \frac12\ \frac12)$ and so on. This, in turn, for $s = 1,\ 3/2,\ 2$ allows one to pass from the first order equations for the reducible representation to second order equations for irrep of improper Poincaré group. For example, for $s = 1$, excluding $f_{1,0}$ and $f_{0,1}$, we obtain

$$m^2 f_{\frac12\frac12}(x, \mathbf{z}) = [\hat p_\mu \hat V^\mu_{12}, \hat p_\nu \hat V^\nu_{21}]_+ f_{\frac12\frac12}(x, \mathbf{z}). \tag{5.81}$$

In general case one also can to pass from the system of first order equations (5.80) on the reducible representation to higher order equations for irrep, for example, to the equations of $1+[s]$ order on the components transforming under irreps $(\frac{s}{2}\ \frac{s}{2})$ or $(\frac{2s+1}{4}\ \frac{2s-1}{4})\oplus(\frac{2s-1}{4}\ \frac{2s+1}{4})$ for the cases of integer or half-integer spin respectively.

Let us consider some particular cases.

For $s=j_1+j_2=1/2$ we have

$$f_{\frac{1}{2}}(x,\mathbf{z})=\chi_\alpha(x)z^\alpha+\overset{*}{\psi}{}^{\dot\alpha}(x)\underline{\overset{*}{z}}_{\dot\alpha}=Z_D\Psi_D(x),\quad \Psi_D(x)=\begin{pmatrix}\chi_\alpha(x)\\ \overset{*}{\psi}{}^{\dot\alpha}(x)\end{pmatrix}, \tag{5.82}$$

where Z_D is given by the formula (5.7). If we substitute (5.82) into equation (5.72) and compare the coefficients at z^α and at $\underline{\overset{*}{z}}_{\dot\alpha}$ in the left and right side, then we obtain the Dirac equation

$$\hat p_\mu\gamma^\mu\Psi_D(x)=m\Psi_D(x),\quad \gamma^\mu=\begin{pmatrix}0 & \sigma^\mu\\ \bar\sigma^\mu & 0\end{pmatrix}. \tag{5.83}$$

According to (5.8) for space inversion we obtain $Z_D\Psi_D(x)\xrightarrow{P}Z_D\Psi_D^P(\bar x)=Z_D\gamma^0\Psi_D(\bar x)$, where $\bar x=(x^0,-x^k)$. The matrix $\gamma^5=\mathrm{diag}\{\sigma^0,-\sigma^0\}$ corresponds to the chirality operator (5.32).

A complex conjugate function corresponds to charge conjugate state,

$$\overset{*}{f}_{1/2}(x,\mathbf{z})=-\psi_\alpha(x)\underline{z}^\alpha-\overset{*}{\chi}{}^{\dot\alpha}(x)\overset{*}{z}_{\dot\alpha},$$

(the minus sign is from anticommutation of spinors, $\psi_\alpha z^\alpha=-z_\alpha\psi^\alpha$) or in the matrix form

$$Z_D\Psi_D(x)\xrightarrow{C}\overset{*}{Z}_D\overset{*}{\Psi}_D(x)=\underline{Z}_D\Psi_D^c(x),\quad \Psi_D^c(x)=-\begin{pmatrix}\psi_\alpha(x)\\ \overset{*}{\chi}{}^{\dot\alpha}(x)\end{pmatrix}=i\sigma^2\begin{pmatrix}\psi^\alpha(x)\\ -\overset{*}{\chi}_{\dot\alpha}(x)\end{pmatrix}, \tag{5.84}$$

where $\underline{Z}_D=(\underline{z}^\alpha,\overset{*}{z}_{\dot\alpha})$ and Z_D obey the same transformation law. Thus we get the different scalar functions to describe particles and antiparticles and hence two Dirac equations for both signs of charge respectively. That matches completely with the results of the article [103]. It was shown there that in the course of a consistent quantization of a classical model of spinning particle namely such (charge symmetric) quantum mechanics appears. At the same time it is completely equivalent to the one-particle sector of the corresponding quantum field theory.

Real functions $f_{1/2}(x,\mathbf{z})=\overset{*}{f}_{1/2}(x,\mathbf{z})$ describing Majorana particle depend on the elements of Z_M (5.9), and correspondingly $\psi^\alpha(x)=-\chi^\alpha(x)=i\sigma^2\chi_\alpha(x)$. Space reflection maps this functions into the functions of $\underline{Z}_M$.

For $s=j_1+j_2=1$ we have

$$f_1(x,\mathbf{z})=\chi_{\alpha\beta}(x)z^\alpha z^\beta+\phi_\alpha{}^{\dot\beta}(x)z^\alpha\underline{\overset{*}{z}}_{\dot\beta}+\psi^{\dot\alpha\dot\beta}(x)\underline{\overset{*}{z}}_{\dot\alpha}\underline{\overset{*}{z}}_{\dot\beta}=\Phi_\mu(x)q^\mu+\frac{1}{2}F_{\mu\nu}(x)q^{\mu\nu}, \tag{5.85}$$

where

$$q^\mu = \frac{1}{2}\sigma^\mu{}_{\alpha\dot\beta} z^\alpha \overset{*}{\underline{z}}{}^{\dot\beta}, \quad q_\mu q^\mu = 0, \qquad q_{\mu\nu} = -q_{\nu\mu} = \frac{1}{2}\left((\sigma_{\mu\nu})_{\alpha\beta} z^\alpha z^\beta + (\bar\sigma_{\mu\nu})_{\dot\alpha\dot\beta} \overset{*}{\underline{z}}{}^{\dot\alpha} \overset{*}{\underline{z}}{}^{\dot\beta}\right), \tag{5.86}$$

$$\Phi_\mu(x) = -\bar\sigma_\mu{}^{\dot\beta\alpha}\phi_{\alpha\dot\beta}(x), \quad F_{\mu\nu}(x) = -2\left((\sigma_{\mu\nu})_{\alpha\beta}\chi^{\alpha\beta}(x) + (\bar\sigma_{\mu\nu})_{\dot\alpha\dot\beta}\psi^{\dot\alpha\dot\beta}(x)\right). \tag{5.87}$$

Substituting (5.85) into equation (5.72), we obtain

$$m\psi^{\dot\alpha\dot\beta}(x) = \frac{1}{2}\hat p_\mu \bar\sigma^{\mu\dot\alpha\gamma}\phi_\gamma{}^{\dot\beta}(x), \quad m\chi_{\alpha\beta}(x) = \frac{1}{2}\hat p_\mu \sigma^\mu{}_{\dot\gamma\alpha}\phi_\beta{}^{\dot\gamma}(x),$$

$$m\phi_\alpha{}^{\dot\beta}(x) = \hat p_\mu(\bar\sigma^{\mu\dot\beta\gamma}\chi_{\alpha\gamma}(x) + \sigma^\mu{}_{\alpha\dot\alpha}\psi^{\dot\alpha\dot\beta}(x)), \tag{5.88}$$

$$mF_{\mu\nu}(x) = \partial_\mu\Phi_\nu(x) - \partial_\nu\Phi_\mu(x), \qquad m\Phi_\mu(x) = \partial^\nu F_{\mu\nu}(x). \tag{5.89}$$

The Duffin–Kemmer equation in the form (5.88) or (5.89) is the equation for irrep $T_{[20]}$ of the 3+2 de Sitter group and thus for the reducible representation $(1\,0) \oplus (\frac{1}{2}\,\frac{1}{2}) \oplus (0\,1)$ of the Lorentz group. This representation contains both four-vector $\Phi_\mu(x)$ and antisymmetric tensor $F_{\mu\nu}(x)$, which correspond to chiralities $\lambda = 0$ and $\lambda = \pm 1$. Excluding components $F_{\mu\nu}(x)$, we obtain second order system only for the components $\Phi_\mu(x)$ transforming under irrep $(\frac{1}{2}\frac{1}{2})$ of the Lorentz group:

$$(\hat p^2 - m^2)\Phi_\mu(x) = 0, \qquad \hat p^\mu \Phi_\mu(x) = 0. \tag{5.90}$$

One can rewrite operator $\hat p_\mu \hat\Gamma^\mu$ in terms of complex variables q^μ and $q^{\mu\nu}$,

$$\hat p_\mu \hat\Gamma^\mu = -i\hat p_\mu (q^{\mu\nu}\partial/\partial q^\nu - q_\nu \partial/\partial q_{\mu\nu}). \tag{5.91}$$

Such conversion to vector indices is possible under considering of any integer spin. Notice that two sets of real spin variables with vector indices can be obtained by the substitution of elements of Z_M and $\underline{Z}_M$ instead of Z_D into (5.86).

One may describe neutral spin 1 field, in particular, by real function of the elements of Majorana z-spinor, $f_1(x, \mathbf{z}) = \overset{*}{f}_1(x, \mathbf{z})$. However, the spaces of quadratic functions of Dirac z-spinor Z_D and Majorana z-spinor Z_M are noninvariant with respect to charge conjugation and space reflection respectively. To describe spin 1 neutral particle coinciding with its antiparticle, one may use bilinear functions of Z_D and $\underline{Z}_D$.

For the cases $s = 1/2$ and $s = 1$ first equation of the system (5.71)-(5.73) (Klein-Gordon equation) is the consequence of other equations. For $s > 1$ the solutions of (5.72) are characterized by spin and mass spectrum, $s_i = \{s,\, s-1, \ldots 1\}$ or $s_i = \{s,\, s-1, \ldots 1/2\}$, $m_i = ms/s_i$. Thus for higher spin fields the Klein-Gordon equation is independent condition, allowing to exclude from spin spectrum all spins, except maximal $s = j_1 + j_2$.

The cases $s = 1/2$ and $s = 1$ are also the exceptions in sense of simplicity of labelling the components by spinor or vector indices. The number of indices of symmetric spin-tensors, necessary for labelling higher spin components, increases in spite of the fact that it is sufficient to use only three operators and therefore only three numbers for labelling of the states belonging to symmetric irreps of $SO(3,2)$.

In particular, for spin 3/2 particle there exist four kinds of components, namely $\psi_{\alpha\beta\gamma}$, $\psi_{\dot\alpha\beta\gamma}$, $\psi_{\dot\alpha\dot\beta\gamma}$, $\psi_{\dot\alpha\dot\beta\dot\gamma}$, corresponding to four possible values of the chirality. For the spin 2 particle the representation in terms of q^μ and $q^{\mu\nu}$ is also cumbersome,

$$f_2(x,q) = \Phi_{\mu\nu}(x)q^\mu q^\nu + \frac{1}{2}F_{\mu\nu,\rho}(x)q^{\mu\nu}q^\rho + \frac{1}{4}F_{\mu\nu,\rho\sigma}(x)q^{\mu\nu}q^{\rho\sigma}, \tag{5.92}$$

with the necessity to fix independent components by means of relations $q_\mu q^\mu = 0$, $q_{\mu\nu}q^\mu + q_{\mu\nu}q^\nu = 0$ and so on.

Thus, beginning from the spin 3/2, it is convenient to use the universal notation $\psi^{m_1 m_2}_{j_1 j_2}(x)$ associated with the decomposition (5.20) over monomial chiral basis (5.22) (see also (B11)-(B13)). Two indices j_1, j_2 label spin $s = j_1 + j_2$ and chirality $\lambda = j_1 - j_2$ and two indices m_1, m_2 label independent components inside the irrep of the Lorentz group. This notation are also suitable for infinite-dimensional representations.

By analogy with 2+1 case, one can find plain wave solutions of the system (5.71)-(5.72) for the any spin s in general form without using matrix representation. Corresponding to the particle moving along x^3 states are eigenstates of the operator $\hat{p}_i\hat{S}^i$ with eigenvalues $|p|\sigma$, where $\sigma = s_3 \mathrm{sign}\, p_3$ is the helicity. These states have the form

$$\begin{aligned} f_{m,s,\sigma}(x,\mathbf{z}) = \ & \textstyle\sum_{\sigma=-s}^{s} C_\sigma e^{ik_0x^0+k_3x^3}(z^1e^a + \overset{*}{\underline{z}}_{\dot{1}}e^{-a})^{s+\sigma}(z^2e^{-a} + \overset{*}{\underline{z}}_{\dot{2}}e^{a})^{s-\sigma} + \\ & \textstyle\sum_{\sigma'=-s}^{s} C_{\sigma'} e^{-ik_0x^0-k_3x^3}(z^1e^a - \overset{*}{\underline{z}}_{\dot{1}}e^{-a})^{s-\sigma}(z^2e^{a} - \overset{*}{\underline{z}}_{\dot{2}}e^{-a})^{s+\sigma}, \end{aligned} \tag{5.93}$$

where e^a is given by (5.76). For the rest particle one can obtain the general solution characterized by the spin projection s' on the direction $\mathbf{n}$ from (5.74) by the rotation $z'_\beta = U_\beta{}^\alpha z_\alpha$, $U \subset SU(2)$. For particle characterized by momentum direction $\mathbf{n}$ and helicity σ, starting from the state (5.93), one can obtain the solution by the analogous rotation.

Improper Poincaré group includes space reflection, which interchanges representations $(j_1\, j_2)$ and $(j_2\, j_1)$. Therefore we consider equations binding these representations (and corresponding components of the solutions of the system (5.71)-(5.73)) more detail.

In the case $j_1 = j_2$ solutions of the system (5.71)-(5.73) are characterized by fixed spin $s = j_1 + j_2$ and mass m. Thus the relations (5.27)-(5.30) are valid, and corresponding $2(2s+1)$ components obey the equations for massive tensor field (5.44).

In general case equations connecting the components transforming under irreps (j_1, j_2) and (j_2, j_1) of the Lorents subgroup have the form

$$\begin{aligned} (2j_2)!(\hat{p}_\mu \hat{V}^\mu_{12})^{2|\lambda|} f_{j_1\, j_2}(x,z,\overset{*}{\underline{z}}) &= (2j_1)! m^{2|\lambda|} f_{j_2\, j_1}(x,z,\overset{*}{\underline{z}}), \\ (2j_1)!(\hat{p}_\mu \hat{V}^\mu_{21})^{2|\lambda|} f_{j_2\, j_1}(x,z,\overset{*}{\underline{z}}) &= (2j_2)! m^{2|\lambda|} f_{j_1\, j_2}(x,z,\overset{*}{\underline{z}}), \end{aligned} \tag{5.94}$$

where $j_1 > j_2$, $|\lambda| = j_1 - j_2$. This set of equations are invariant under space reflection. Using the decomposition (5.79) and explicite form of the general solution (5.74) of the system (5.71)-(5.73) in the rest frame, one can prove the validity of (5.94) by direct calculation. Going over to spin-tensor notation, we get

$$\begin{aligned} p_\mu \bar{\sigma}^{\mu\dot{\alpha}_{2j_2+1}\alpha_{2j_2+1}} \ldots p_\nu \bar{\sigma}^{\nu\dot{\alpha}_{2j_1}\alpha_{2j_1}} \psi^{\dot{\alpha}_1\ldots\dot{\alpha}_{2j_2}}_{\alpha_1\ldots\alpha_{2j_1}}(x) &= m^{2|\lambda|}\psi^{\dot{\alpha}_1\ldots\dot{\alpha}_{2j_1}}_{\alpha_1\ldots\alpha_{2j_2}}(x), \\ p_\mu \sigma^\mu{}_{\alpha_{2j_2+1}\dot{\alpha}_{2j_2+1}} \ldots p_\nu \sigma^\nu{}_{\alpha_{2j_1}\dot{\alpha}_{2j_1}} \psi^{\dot{\alpha}_1\ldots\dot{\alpha}_{2j_1}}_{\alpha_1\ldots\alpha_{2j_2}}(x) &= m^{2|\lambda|}\psi^{\dot{\alpha}_1\ldots\dot{\alpha}_{2j_2}}_{\alpha_1\ldots\alpha_{2j_1}}(x). \end{aligned} \tag{5.95}$$

Equations (5.95) are consequences of the system (5.71)-(5.73), but unlike this system in general case require some supplementary conditions to fix mass and spin.

Equations (5.95) are the first order equations only in the case $|\lambda| = 1/2$, which corresponds to representations $(\frac{2j\pm1}{4}\, \frac{2j\mp1}{4})$, $j = j_1 + j_2$, describing half-integer spins. In this case,

going over to vector indices and supplementing the equations by subsidiary conditions (5.45) (which also are the consequences of the system (5.71)-(5.73) and exclude components with $s < j_1 + j_2$), we obtain the Rarita–Schwinger equations [64]

$$(\hat{p}_\mu \gamma^\mu - m)\Psi_{\mu_1\mu_2\dots\mu_n}(x) = 0, \quad \gamma^\mu \Psi_{\mu\mu_2\dots\mu_n}(x) = 0, \tag{5.96}$$

where $n = 2s - 1$ and $\Psi_{\mu_1\dots\mu_n}(x)$ is a four-component column composed of $\Psi_{\mu_1\dots\mu_n\alpha}(x)$ and $\Psi_{\mu_1\dots\mu_n}{}^{\dot\alpha}(x)$. The conditions $\partial^\mu\Psi_{\mu\mu_2\dots\mu_n}(x) = 0$ and $\Psi^\mu{}_{\mu\dots\mu_n}(x) = 0$ appear as consequences of these two equations [21].

A case $|\lambda| = s$ corresponding to representations $(s\,0)$ and $(0\,s)$ is preferred because of minimal number of components. In this case the equations (5.95) are $2s$ order Joos-Weinberg equations [104–106] of so called $2(2s+1)$-component theory,

$$\begin{aligned} &p_\mu \bar\sigma^{\mu\dot\alpha_1\alpha_1} \dots p_\nu \bar\sigma^{\nu\dot\alpha_{2s}\alpha_{2s}} \psi_{\alpha_1\dots\alpha_{2s}}(x) = m^{2s}\psi^{\dot\alpha_1\dots\dot\alpha_{2s}}(x), \\ &p_\mu \sigma^\mu{}_{\alpha_1\dot\alpha_1} \dots p_\nu \sigma^\nu{}_{\alpha_{2s}\dot\alpha_{2s}} \psi^{\dot\alpha_1\dots\dot\alpha_{2s}}(x) = m^{2s}\psi_{\alpha_1\dots\alpha_{2s}}(x). \end{aligned} \tag{5.97}$$

In the rest frame as a consequence we obtain $p_0^{4s} = m^{4s}$ and for $s \geq 1$ Joos-Weinberg equations have solutions with complex energy p_0, $|p_0| = m$. The existence of such solutions was pointed out also in [107].

D. Relativistic wave equations, invariant under improper Poincaré group. Equations for several scalar functions

Above we have considered the linear equations for *one* scalar function on the group. The condition of invariance under space reflection leads us to the system (5.71)-(5.73) for particle with spin $s = j_1 + j_2$ and mass m.

For the construction of invariant wave equations one may also use the operators $\hat{p}_\mu \hat{V}^\mu_{ik}$, which are not invariant under space reflections. Using several scalar functions $f(x, \mathbf{z})$, it is possible to restore the invariance under space reflections.

In particular, equations (5.55) containing operators $\hat{V}^\mu_{12(k)}$ and $\hat{V}^\mu_{21(k)}$ interlock two scalar functions. Using the decomposition (5.23) in terms of spin-tensors, we obtain Dirac–Fierz–Pauli equations [63,48],

$$\begin{aligned} &\hat{p}_\mu \bar\sigma^{\mu\dot\alpha\beta} \psi^{\dot\alpha_1\dots\dot\alpha_l}_{\beta\,\beta_1\dots\beta_n}(x) = \varkappa \psi^{\dot\alpha\,\dot\alpha_1\dots\dot\alpha_l}_{\beta_1\dots\beta_n}(x) \\ &\hat{p}_\mu \sigma^\mu{}_{\beta\dot\alpha} \psi^{\dot\alpha\,\dot\alpha_1\dots\dot\alpha_l}_{\beta_1\dots\beta_n}(x) = \varkappa \psi^{\dot\alpha_1\dots\dot\alpha_l}_{\beta\,\beta_1\dots\beta_n}(x). \end{aligned} \tag{5.98}$$

These equations connect two functions transforming under irreps $(\frac{n}{2} + \frac{1}{2}\,\frac{l}{2})$ and $(\frac{n}{2}\,\frac{l}{2} + \frac{1}{2})$ of the Lorentz group and at $n = l$ mapping to one another under parity transformation.

Let us consider a system of equations of the form (5.55),(5.56), which connect several scalar functions with different j_1, j_2. The equations of this system interlock the representation (j_1, j_2) with at least one of the representations $(j_1 \pm 1, j_2 \mp 1)$, $(j_1 \pm 1, j_2 \pm 1)$. This allows one to identify this system with general Gel'fand–Yaglom equations [67,59]

$$(\alpha^\mu \hat{p}_\mu - \varkappa)\psi = 0, \quad [S^{\lambda\mu}, \alpha^\nu] = i(\eta^{\mu\nu}\alpha^\lambda - \eta^{\lambda\nu}\alpha^\mu). \tag{5.99}$$

In the present approach the latter relation is a consequence of the commutation relations $[\hat{S}^{\lambda\mu}, \hat{V}^{\nu}_{ik}] = i(\eta^{\mu\nu}\hat{V}^{\lambda}_{ik} - \eta^{\lambda\nu}\hat{V}^{\mu}_{ik})$. This relation is necessary for Poincaré invariance of the equation [16,59].

Supplemented by commutation relations $[\alpha^\mu, \alpha^\nu] = S^{\mu\nu}$ finite-component equations of the form (5.99) are known as Bhabha equations [56], although for the first time they were systematically considered by Lubanski [55]. These equations are classified according to the finite-dimensional irreps of the 3+2 de Sitter group $SO(3,2)$. Other possible commutation relations of matrices α^μ are discussed in [108].

Considered above equation (5.72) on a scalar function is the particular case of Bhabha equations. This case correspond to symmetric irreps $T_{[2s\,0]}$ of the 3+2 de Sitter group. Generally speaking, the Bhabha equations are characterized by finite number of different m and s. Therefore, this equations connect the fields transforming under nonequivalent irreps of the Poincaré group.

If equations include the operators $\hat{p}_\mu\hat{V}^{\mu}_{11}$ and $\hat{p}_\mu\hat{V}^{\mu}_{22}$, then either the equations describe at least two different spins s or the condition $s = j_1 + j_2$ connecting spin s with a highest weight of irrep of Lorentz group is not valid.

Cite as an example the system interlocking irreps (00) and $(\frac{1}{2}\frac{1}{2})$ of the Lorentz group:

$$\hat{p}_\mu\hat{V}^{\mu}_{11}f_{00}(x,\mathbf{z}) = \varkappa_1 f_{\frac{1}{2}\frac{1}{2}}(x,\mathbf{z}), \quad \hat{p}_\mu\hat{V}^{\mu}_{22}f_{\frac{1}{2}\frac{1}{2}}(x,\mathbf{z}) = \varkappa_2 f_{00}(x,\mathbf{z}) \tag{5.100}$$

where $f_{00}(x,\mathbf{z}) = \psi(x)$, $f_{\frac{1}{2}\frac{1}{2}}(x,\mathbf{z}) = \psi_\alpha{}^{\dot\beta}(x)z^\alpha \overset{*}{\underline{z}}_{\dot\beta}$; in component-wise form we have $\hat{p}_\mu\psi = 2\varkappa_1\psi_\mu$, $\hat{p}_\mu\psi^\mu = \varkappa_2\psi$. In the rest frame one may obtain $\varkappa_2 = 2\varkappa_1 = m$. Thus the system (5.100) is equivalent to Duffin equation for scalar particles, which correspond to five-dimensional vector irrep $T_{[01]}$ of $SO(3,2)$ group.

E. Relativistic wave equations, invariant under improper Poincaré group. Equations for particles with composite spin

Many-particle systems are described by the functions of the sets of variables $x_{(i)}, z_{(i)}, \overset{*}{\underline{z}}_{(i)}$. But here we will consider not many-particle systems in usual sense, but some objects corresponding to functions $f(x, z_{(1)}, \overset{*}{\underline{z}}_{(1)}, \ldots, z_{(n)}, \overset{*}{\underline{z}}_{(n)})$, (or, briefly, $f(x, \{\mathbf{z}_{(i)}\})$), i.e. to functions of one set of x and several sets of $\mathbf{z}$. These objects one may interpret as particles with composite spin.

As an example we will consider the Ivanenko-Landau-Kähler (or Dirac-Kähler) equation [109,110]. Let us write some linear in $\mathbf{z}_{(1)}$ and $\mathbf{z}_{(2)}$ scalar function $f(x, \mathbf{z}_{(1)}, \mathbf{z}_{(2)})$ in the form

$$f(x,\mathbf{z}_{(1)},\mathbf{z}_{(2)}) = Z^{(1)}_D\Psi(x)(Z^{(2)}_D)^\dagger = \sum_{i,j=1}^{4}(Z^{(1)}_D)_i\Psi_{ij}(x)(\overset{*}{Z}{}^{(2)}_D)_j, \tag{5.101}$$

where $Z_D = (z^1\, z^2\, \overset{*}{\underline{z}}_{\dot 1}\, \overset{*}{\underline{z}}_{\dot 2})$, and $\Psi(x)$ is a 4×4 matrix with a transformation rule

$$\Psi'(x') = \check{U}\Psi(x)(\check{U})^\dagger, \quad \check{U} = \mathrm{diag}\{U, (U^{-1})^\dagger\},$$

in contrast to the transformation rule $\Psi'_D(x') = \check{U}\Psi_D(x)$ of Dirac spinor (5.83). Let us impose the equation on the first ("left") spin subsystem,

$$(\hat{p}_\mu \hat{\Gamma}^\mu_{(1)} - m/2) f(x, \mathbf{z}_{(1)}, \mathbf{z}_{(2)}) = 0, \tag{5.102}$$

and do not impose any conditions on the second (”right”) spin subsystem. Writing (5.102) in component-wise form, one can obtain Ivanenko-Landau-Kähler equation in spinor matrix representation

$$(\hat{p}_\mu \gamma^\mu - m)\Psi(x) = 0. \tag{5.103}$$

According to (5.103), 16 components $\Psi_{ij}(x)$ obey Klein-Gordon equation, therefore mass is equal to m. Spin of both subsystems is equal two 1/2. The spin of system is indefinite, and there are both spin 0 and spin 1 components.

The consideration of this equation is associated mainly with the attempts to describe fermions by the antisymmetric tensor fields (see, for example, [22,111–113] and also [114] as a good introduction). The spin subsystems (”left-spin” and ”right-spin”) were considered in [111,113]).

Let us consider now linear symmetric functions of $\mathbf{z}_{(1)}, \ldots, \mathbf{z}_{(n+l)}$:

$$f_{\frac{n}{2}\frac{l}{2}}(x, \{\mathbf{z}_{(i)}\}) = \psi^{\dot\alpha_1,\ldots,\dot\alpha_l}_{\beta_1,\ldots,\beta_n}(x) \sum z^{\beta_1}_{(1)} \ldots z^{\beta_n}_{(n)} \overset{*}{\underline{z}}_{(n+1)\dot\alpha_1} \ldots \overset{*}{\underline{z}}_{(n+l)\dot\alpha_l}, \tag{5.104}$$

where symmetric spinors $\psi^{\dot\alpha_1,\ldots,\dot\alpha_l}_{\beta_1,\ldots,\beta_n}(x)$ transform under irreps $(n/2, l/2)$, and all permutations of $1, \ldots, n+l$ are summed over. As a consequence of symmetry of the multispinors with respect to index permutations, spin subsystems are indistinguishable, and this allow us to use functions of several sets of spin variables for describing usual particles.

So, one may obtain Dirac–Fierz–Pauli equations (5.98), acting by the operators $\hat{V}^\mu_{12(k)}$ and $\hat{V}^\mu_{21(k)}$ on the functions (5.104) corresponding to irreps $(\frac{n}{2}+\frac{1}{2}, \frac{l}{2})$ and $(\frac{n}{2}, \frac{l}{2}+\frac{1}{2})$ of the Lorentz group:

$$\begin{aligned} \hat{V}^\mu_{12(k)} f_{\frac{n}{2}+\frac{1}{2},\frac{l}{2}}(x, \{\mathbf{z}_{(i)}\}) &= \varkappa f_{\frac{n}{2},\frac{l}{2}+\frac{1}{2}}(x, \{\mathbf{z}_{(i)}\}), \\ \hat{V}^\mu_{21(k)} f_{\frac{n}{2},\frac{l}{2}+\frac{1}{2}}(x, \{\mathbf{z}_{(i)}\}) &= \varkappa f_{\frac{n}{2}+\frac{1}{2},\frac{l}{2}}(x, \{\mathbf{z}_{(i)}\}). \end{aligned} \tag{5.105}$$

In general case a linear symmetric function of $\mathbf{z}_{(k)}$, $k = 1, \ldots, 2j$ has the form

$$f_j(x, \{\mathbf{z}_{(i)}\}) = \sum_{n,l;n+l=2j} \psi^{\dot\alpha_1 \ldots \dot\alpha_l}_{\beta_1 \ldots \beta_n}(x) z^{\beta_1}_{(1)} \ldots z^{\beta_n}_{(n)} \overset{*}{\underline{z}}_{(n+1)\dot\alpha_1} \ldots \overset{*}{\underline{z}}_{(n+l)\dot\alpha_l}. \tag{5.106}$$

Functions (5.106) correspond to symmetric part of the representation $\left((\frac{1}{2}\,0) \oplus (0\,\frac{1}{2})\right)^{2j}$. This symmetric part expand into direct sum of irreps $(j_1\,j_2)$, $j_1 + j_2 = j$. Impose on each spin subsystem the condition

$$(\hat{p}_\mu \hat{\Gamma}^\mu_{(k)} - m/2) f(x, \mathbf{z}_{(1)}, \ldots, \mathbf{z}_{(2j)}) = 0, \quad k = 1, \ldots, 2j. \tag{5.107}$$

Rewriting this equations in four-component form, we obtain Bargmann–Wigner equations [2,21,115]

$$(\hat{p}_\mu \gamma^\mu_{(k)} - m)_{\alpha_k \beta_k} \psi_{\beta_1 \ldots \beta_k \ldots \beta_{2j}}(x) = 0. \tag{5.108}$$

As a consequence of (5.107), one may obtain equations for system as a whole:

$$(\hat{p}^2 - m^2) f(x, \mathbf{z}_{(1)}, \ldots, \mathbf{z}_{(2j)}) = 0,$$
$$(\hat{p}_\mu \hat{\Gamma}^\mu - ms) f(x, \mathbf{z}_{(1)}, \ldots, \mathbf{z}_{(2j)}) = 0, \quad \hat{\Gamma}^\mu = \sum \hat{\Gamma}^\mu_{(k)}, \tag{5.109}$$

which are analogous with equations (5.71)-(5.72) at the case $s = j_1 + j_2$. Both Bargmann–Wigner equations and system (5.109) have $2(2s+1)$ independent solutions $\psi(x)$, and therefore this systems are equivalent.

F. Relativistic wave equations: comparative consideration

In the framework of the group-theoretical classification of the scalar fields $f(x, \mathbf{z})$ on the Poincaré group we have obtained two types of equations describing unique spin and mass, namely equations for the eigenfunctions of the Casimir operator of the Lorentz spin subgroup (j_1 and j_2 are fixed, see (5.30)) and equations for the eigenfunctions of the Casimir operator of the $SO(3,2)$ group (the sum $j_1 + j_2$ is fixed). Below we will consider comparative characteristics of these equations and also the case $(j_1\, j_2) \oplus (j_2\, j_1)$ corresponding to irreps of the improper Poincaré group but requiring two scalar functions for its formulation.

1. Equations for the functions corresponding to the fixed irrep $(j_1\, j_2)$ of the Lorentz group. Mass and spin irreducibility conditions leave $2(2s+1)$ independent components corresponding to two improper Poincaré group irreps differed by sign of p_0. For $s = j_1 + j_2$ the equations in spin-tensor form constitute the system of the Klein-Gordon equation and the subsidiary condition (5.42), which eliminates components with other possible values of spin s for fixed j_1, j_2, $|j_1 - j_2| \le s < j_1 + j_2$. For $s \ne j_1 + j_2$ one should consider general subsidiary condition (5.43). An alternative to the use of subsidiary condition is the consideration of functions of momentum and spin variables with invariant constraints (5.39).

There exist two preferred cases. The first corresponds to the representations $(\frac{s}{2}\, \frac{s}{2})$ mapping onto themselves under space reflection and are most often used to describe integer spins. The second corresponds to the representations $(s\, 0)$ and $(0\, s)$. In this case there is not necessity to impose subsidiary conditions since they are fulfilled identically.

2. Equations for the functions corresponding to the representations $(j_1\, j_2)$ and $(j_2\, j_1)$, $j_1 \ne j_2$, which are interchanged under space reflection. Unlike the considered above equations for fixed j_1, j_2, these equations in general case do not assume formulation as equations for one scalar function $f(x, \mathbf{z})$. The conditions of mass and spin irreducibility leave $4(2s+1)$ independent components corresponding to four improper Poincaré group irreps differed by sign of p_0 and intrinsic parity η. To choose $2(2s+1)$ components corresponding to fixed sign of η or $p_0\eta$ it is necessary to supplement these conditions by equations (5.94) connecting components corresponding to $(j_1\, j_2)$ and $(j_2\, j_1)$.

Equations (5.94) are first order equations only for the representations $(j+\frac{1}{2}\; j) \oplus (j\; j+\frac{1}{2})$. These representations and associated with them Rarita–Schwinger equations (5.96) are most often used to describe half-integer spins. However, just as in the case of representations $(j\, j)$, subsidiary conditions supplement the field equations, and the number of equations exceeds the number of field components. Therefore one has an overdetermined set of equations which, although consistent in the free-field case, for $s > 1$ becomes self-contradictory with minimal electromagnetic coupling [48]. In order to avoid inconsistency it is possible to give a Lagrangian formulation, introducing auxiliary fields [48–50], but this formulation leads to acasual propagation with minimal electromagnetic coupling [51,52,116–119].

For the case $(s\,0)\oplus(0\,s)$ one can construct 2(2s+1)-component theory, but corresponding Joos-Weinberg equations of $2s$ order [104,105] (see (5.97)) for $s \geq 1$ have also solutions with complex energy.

The second order equation for representation $(s\,0)\oplus(0\,s)$, $(\hat{P}^2 - \frac{e}{2s}\hat{S}^{\mu\nu}F_{\mu\nu} - m^2)\psi(x) = 0$, [120,121,53,122] for free particle possesses $4(2s+1)$ independent components differed by spin projection and by signs of p_0 and η. On the other hand, this equation describes unique mass and spin and is characterized by casual solutions. In particular, exact solutions in external constant uniform electromagnetic field are known [54]. One may rewrite the above equation as first order equation with minimal coupling for representations $(s\,0) \oplus (s-\frac{1}{2}\;\frac{1}{2}) \oplus (\frac{1}{2}\;s-\frac{1}{2}) \oplus (0\,s)$. As noted in [116], this is the simplest class of describing unique mass and spin representations, which led to first order equations without subsidiary conditions.

3. Equations (5.71)-(5.72) for eigenfunctions of the Casimir operator (5.73) of $SO(3,2)$ group with eigenvalues $4s(s+2)$, $s = j_1 + j_2$:

$$(\hat{p}_\mu \hat{\Gamma}^\mu - ms) f(x, z, \underline{\overset{*}{z}}) = 0, \quad (\hat{p}^2 - m^2) f(x, z, \underline{\overset{*}{z}}) = 0. \tag{5.110}$$

The condition of spin irreducibility (5.51) is a consequence of this system.

The first equation of the system is the Bhabha equation [55,56] corresponding to symmetric irrep $T_{[2s\,0]}$ of the group $Sp(4,R) \sim SO(3,2)$. This equation represent a straightforward higher spins generalization of Dirac and spin 1 Duffin–Kemmer equations. Both Bhabha equations and the problem of minimal coupling for these equations were most detail considered in papers of Krajcik and Nieto (see [57]; it contains references to the six earlier papers). The theory is casual with minimal electromagnetic coupling [71], but in general case Bhabha equations describe multi-mass systems. Notice that the connection of the Rarita–Schwinger and Bargmann–Wigner equations with Bhabha equations was considered also in [123].

The solutions of the system (5.110) have the components transforming under $2s+1$ irreps (j_1, j_2), $j_1 + j_2 = s$, of the Lorentz group. But the components corresponding to different chiralities $\lambda = j_1 - j_2$ are not independent. In contrast to left generators of the Poincaré group operators $\hat{\Gamma}_\mu$ do not commute with chirality operator (which is the right generator of the Poincaré group) and combine $2s+1$ representations of the Lorentz group into one irrep of the 3+2 de Sitter group $SO(3,2)$.

The current component j^0 is positive definite for half-integer spin particles and the energy density is positive definite for integer spin particles, see Appendix B.

In the rest frame equations (5.110) have $2s+1$ positive- and $2s+1$ negative-frequency solutions labelled by different spin projections, see (5.74), and half-integer spins solutions with opposite frequency are characterized by opposite parity. In the ultrarelativistic limit two solutions with opposite sign of p_0 correspond to any of $2s+1$ of possible values of chirality, see (5.78).

Thus the system (5.110) describes a particle with unique spin and mass, is invariant under parity transformation and possesses $2(2s+1)$ independent components.

Let us briefly consider the problem of equivalence of the different RWE. In the case of free fields, using the relation

$$[\partial_\mu, \partial_\nu] = 0, \tag{5.111}$$

one can establish the equivalence of wide class of RWE.

So, as we have established above, in a free case the system (5.110) and the Bargmann–Wigner equations (5.107), which both describe a particle by means wave functions with components transforming under $2s+1$ irreps $(j_1\, j_2)$, $j_1 + j_2 = s$, of the Lorentz group, are equivalent. However, the formulation (5.110) is more general since unlike Bargmann–Wigner equations can be considered also in the case of infinite-dimensional unitary representations of the Lorentz group, as was done above with an analogous system in 2+1-dimensional case.

The considered above free equations for representations $(j_1\, j_2)$ or $(j_1\, j_2) \oplus (j_2\, j_1)$ can be obtained as a consequence of the Bargmann–Wigner equations [21,115] or the system (5.110) by excluding of other components. In general case for $m \neq 0$ one may express all components in terms of the components corresponding to two chiralities $\pm\lambda$, where $-s \leq \lambda \leq s$.

It is obvious that the coupling, which is minimal for one system, is not minimal for another "equivalent" system if one uses the relation (5.111) to prove this equivalence in a free case. These equations will differ by the terms proportional to the commutator of covariant derivatives $[D_\mu, D_\nu] = igF_{\mu\nu}$.

Therefore, when an interaction is introduced, the system of equations can be found inconsistent if, taking account of (5.111), some equations are the consequences of another. In particular, spin 1 Bargmann–Wigner equations with minimal electromagnetic coupling are inconsistent (for the prove see, for example, [124]), but equivalent to them in a free case Duffin–Kemmer and Proca equations with minimal coupling are consistent and characterized by casual solutions [117].

Recently different approaches to introduce interactions for higher spin massive fields have been considered (see, in particular, [54,125–127]). Keeping in mind the present approach, we hope that new possibilities to describe interacting higher spin fields will arise.

VI. EQUATIONS FOR FIXED SPIN AND MASS: GENERAL FEATURES

Consider now the general properties of the obtained equations describing a particle with unique mass $m > 0$ and spin s in two dimensions

$$\hat{p}^2 f(x,\theta) = m^2 f(x,\theta), \tag{6.1}$$

$$\hat{p}_\mu \hat{\Gamma}^\mu f(x,\theta) = ms f(x,\theta), \tag{6.2}$$

in three dimensions

$$\hat{p}^2 f(x,\mathbf{z}) = m^2 f(x,\mathbf{z}), \tag{6.3}$$

$$\hat{p}_\mu \hat{S}^\mu f(x,\mathbf{z}) = ms f(x,\mathbf{z}), \tag{6.4}$$

$$\hat{S}_\mu \hat{S}^\mu f(x,\mathbf{z}) = S(S+1) f(x,\mathbf{z}), \tag{6.5}$$

in four dimensions

$$\hat{p}^2 f(x,\mathbf{z}) = m^2 f(x,\mathbf{z}), \tag{6.6}$$

$$\hat{p}_\mu \hat{\Gamma}^\mu f(x,\mathbf{z}) = ms f(x,\mathbf{z}), \tag{6.7}$$

$$\hat{S}_{ab} \hat{S}^{ab} f(x,\mathbf{z}) = 4S(S+2) f(x,\mathbf{z}). \tag{6.8}$$

In the latter case in addition we suppose $s = \pm S$ to avoid nontrivial spin and mass spectrum.

In all dimensions the first equation (condition of the mass irreducibility) is the eigenvalue equation for the Casimir operator of the Poincaré group. But the other equations, although seem similar, have different origin in even and odd dimensions. This is related to the different role of space inversion.

In 2+1 dimensions other equations (6.4)-(6.5) are eigenvalue equations for the Casimir operator of the Poincaré group and the spin Lorentz subgroup.

In even dimensions the Casimir operators of the Lorentz subgroup do not commute with the space inversion operator, and space inversion combines two labelled by chiralities $\pm\lambda$ equivalent representations of the proper Poincaré group into representation of the improper Poincaré group. If one rejects equations, which fix chirality (in 3+1 dimensions this corresponds to the transition to the system (5.50)-(5.52)), then in the rest frame it is easy to see that there is redundant number of independent components. Thus it is necessary to construct equation, binding the states with different chiralities, and correspondingly new set of commuting operators. It can be down by using supplementary operators $\hat{\Gamma}^\mu$, which extend Lorentz group $SO(D,1)$ up to $SO(D,2)$ group with the maximal compact subgroup $SO(D)\otimes SO(2)$. Operator $\hat{\Gamma}^0$ is the generator of compact $SO(2)$ subgroup.

Third equation of the system fixes the power $2S$ of homogeneity of the functions $f(x,\mathbf{z})$ in $\mathbf{z}$ and therefore fixes irrep of the Lorentz group in 2+1 dimensions or of the 3+2 de Sitter group in 3+1 dimensions. (In 1+1 dimensions there exists analogous equation $\hat{\Gamma}_a\hat{\Gamma}^a f(x,\theta) = s(s+1)f(x,\theta)$, but, in fact, this equation defines the structure of $\hat{\Gamma}^\mu$.)

Positive (half-)integer $S=s$ correspond to finite-dimensional nonunitary irreps of the Lorentz (or de Sitter) group, which realize in the space of the power $2s$ polynomials in $\mathbf{z}$.

Negative $S=-s$ correspond to infinite-dimensional unitary irreps. The unitary property allows one to combine probability amplitude interpretation and relativistic invariance (the desirability of this combination was stressed by Dirac in [79]). Thus the equations under consideration allow two approaches to the description of the same spin by means of both finite-dimensional nonunitary and infinite-dimensional unitary irreps.

In 1+1 and 2+1 dimensions there is the possibility of the existence of particles with fractional spin since the groups $SO(1,1)$ and $SO(2,1)$ do not contain compact Abelian subgroup. However, the description of massive particles with fractional spin can be given only in terms of the infinite-dimensional irreps of the group $SO(2,1)$. This is another reason to consider infinite-dimensional irreps.

Fixing the irrep of the Lorentz (or de Sitter) group with the help of the third equation of the system, one can come to usual multicomponent matrix description by the separation of space and spin variables: $f(x,\mathbf{z}) = \sum \phi_n(\mathbf{z})\psi_n(x)$, where $\phi_n(\mathbf{z})$ form the basis in the representation space of the Lorentz (or de Sitter) group. Thus, depending on the choice of the solution of the third equation, second equation in matrix representation is either finite-component equation or infinite-component equation of Majorana type.

For fundamental spinor irreps the action of differential operators $2\hat{S}^\mu$ in 2+1 dimensions and $2\hat{\Gamma}^\mu$ in 1+1 and 3+1 dimensions in the space of functions $f(x,\mathbf{z})$ on the Poincaré group can be rewritten in terms of action of corresponding γ-matrices on the functions $\psi(x)$.

Differential operators $\hat{\Gamma}^\mu$ and matrices $\gamma^\mu/2$ obey the same commutation relations

$$[\hat{\Gamma}^\mu,\hat{\Gamma}^\nu] = -i\hat{S}^{\mu\nu}, \quad [\hat{S}^\mu,\hat{S}^\nu] = -i\epsilon^{\mu\nu\rho}\hat{S}_\rho.$$

In 3+1 dimensions operators $\hat{\Gamma}^\mu$ and $\hat{S}^{\mu\nu}$ obey the commutation relations of generators of

$SO(3,2)$ group, see (5.62).

Anticommutation relations for operators $\hat{S}^\mu$ in 2+1 and $\hat{\Gamma}^\mu$ in 1+1 and 3+1 dimensions are analogous with the relations for γ-matrices,

$$[\hat{S}^\mu, \hat{S}^\nu]_+ = \frac{1}{2}\eta^{\mu\nu}, \quad [\hat{\Gamma}^\mu, \hat{\Gamma}^\nu]_+ = \frac{1}{2}\eta^{\mu\nu},$$

and are valid only for fundamental spinor irreps. This is group-theoretical property connected with the fact that for these irreps the double action of lowering or raising operators on any state gives zero as a result. (Notice that, besides the case of spinor irreps of orthogonal groups, anticommutation relations also take place for fundamental N-dimensional irreps of $Sp(N)$ and $SU(N)$ groups [128].)

For $s = 1/2$ and $s = 1$ the first equation of the system (condition of mass irreducibility) is a consequence of (6.4) or (6.7). In general case the second equation of the system describes multi-mass systems $m_i s_i = ms$. Thus for $s > 1$ it is necessary to consider both equations.

Consider some characteristics of the equations associated with finite-dimensional irreps of the Lorentz group. If we reject the first equation of the system (i.e. the condition of mass irreducibility), then for the second equation of the system the component j^0 of the current vector is positive definite only for $s = 1/2$, and the energy density $-T^{00}$ (see (4.27)) is positive definite only for $s = 1$. (The case $s = 1$ in 3+1 dimensions has considered in detail in [59,129]). However, for the system as a whole the component j^0 of the current vector is positive definite for any half-integer spin, and energy density is positive definite for any integer spin. Besides, in the rest frame half-integer spin solutions with opposite sign of p_0 are characterized by opposite parity.

For the case of infinite-component equations in 2+1 dimensions, the energy is positive definite for any spin, and j^0 is positive or negative definite in accordance with the sign of charge.

The consideration of the field on the Poincaré group also allows one to ensure essential progress in the problem of practical computations for multicomponent equations. As was noted in [68], the general investigation of Gel'fand–Yaglom equations "revealed a number of interesting features, but ... the use of such equations (or more accurately, systems of a large or infinite number of equations) for any practical computations is not possible". In the present approach, due to the use of spin differential operators instead of finite or infinite-dimensional matrices, from the technical point of view there is no essential distinction in the consideration of the equations associated with various finite-dimensional and infinite-dimensional representations of the Lorentz group. Therefore the present approach is adequate to work with higher spins and positive energy wave equations. For example, the use of spin variables $\mathbf{z}$ has allowed us to obtain explicit compact form of general plane wave solutions for any spin (including fractional spin in 2+1 dimensions).

Notice that unlike the equations for particles with unique mass and spin, in general case RWE with mass and spin spectrum can either interlock several scalar functions $f(x, \mathbf{z})$ (as general Gel'fand–Yaglom equations and, in particular, Bhabha equations) or describe objects with composite spin, which correspond to the functions $f(x, \mathbf{z}_{(1)}, \ldots \mathbf{z}_{(n)})$ of one set of space-time coordinates x and several sets of spin coordinates $\mathbf{z}$ (as Ivanenko-Landau-Kähler or Dirac-Kähler equation).

VII. CONCLUSION

In this paper we have elaborated a general scheme of analysis of fields on the Poincaré group and have applied it in two, three and four-dimensional cases.

Considering the left GRR of the Poincaré group, we introduce the scalar field $f(x, \mathbf{z})$ on the group, where x are coordinates in Minkowski space and $\mathbf{z}$ are coordinates on the Lorentz group. The connection between the left GRR and the scalar field allows one to use the powerful mathematical method of harmonic analysis on a group, at the same time supporting the consideration by physical motivations.

The consideration of the functions $f(x, \mathbf{z})$ guarantees the possibility to describe arbitrary spin particles because any irrep of a group is equivalent to some sub-representation of GRR. Thus we deal with an unique field containing all masses and spins. As a consequence, we have:

1. The explicit form of spin projection operators does not depend on the spin value. These operators are the differential operators with respect to $\mathbf{z}$.

2. For this scalar field and thus for arbitrary spin discrete transformations C, P, T are defined as the automorphisms of the Poincaré group.

3. RWE arise under the classification of the functions on the Poincaré group by eigenvalues of invariant operators and have the same form for arbitrary spin.

The switch to the usual multicomponent description by functions $\psi_n(x)$ corresponds to a separation of the space-time and spin variables, $f(x, \mathbf{z}) = \sum \phi_n(\mathbf{z})\psi_n(x)$, where $\phi_n(\mathbf{z})$ and $\psi_n(x)$ transform under contragradient representations of the Lorentz group. The use of the transformation rules of $x, \mathbf{z}$ under automorphisms enables us to deduce the transformation rules of $\psi_n(x)$ under C, P, T without any consideration of the specific form of equations of motion.

It is shown that in even dimensions the consistent consideration of invariant with respect to space reflection RWE requires to use the generators of group $SO(D, 2)$, which is an extension of the corresponding Lorentz group $SO(D, 1)$.

The interpretation of the right generators belonging to the complete set of commuting operators on the Poincaré group is given. This interpretation is similar to Wigner and Casimir interpretation of right generators of the rotation group in the nonrelativistic theory (see [24,26]). Like in the nonrelativistic case, right generators define some quantum numbers, which do not depend on the choice of the laboratory frame. In particular, in 3+1-dimensional case three right generators of the Poincaré group define Lorentz characteristics j_1, j_2 and chirality, and fourth right generator distinguishes particles and antiparticles.

Using complete sets of the commuting operators on the group, we classify scalar functions $f(x, \mathbf{z})$. As one of the results of this classification we reproduce essentially all known finite-component RWE. Moreover, such an approach allows one to consider some alternative possibilities, which have not been formulated before. In particular, in 3+1-dimensional case we write out general subsidiary conditions (5.43) corresponding to $s \neq j_1 + j_2$. On the other hand, instead of subsidiary conditions one may consider functions of momentum p and spin variables $\mathbf{z}$ with invariant constraints (5.39). It is shown that the set of operators related to higher spin equations in 3+1 dimensions obeys commutation relations of $so(3, 3)$ algebra, which coincide with the algebra of γ-matrices for spin 1/2. But unlike the latter case the set of operators for higher spin equations is not closed with respect to anticommutation.

In the framework of the classification of scalar functions we get also positive energy wave equations allowing probability amplitude interpretation and associated with infinite-dimensional unitary representations of the Lorentz group. Along with the alternative description of integer or half-integer spin fields, just these equations ensure description of fractional spin fields in 1+1 and 2+1 dimensions.

The consideration of the scalar field on the Poincaré group have allowed us both to obtain new results and to reproduce the main results of RWE theory, which earlier were obtained by means of different reasons and methods. Thus a general approach to the construction of different types of RWE is established. Besides, one may consider this approach as an alternative method to construct a detail theory of the Poincaré group representations.

Notice that the approach under consideration can be directly applied to higher dimensional cases and, as we hope, can be generalized to other space-time symmetry groups, such as de Sitter and conformal groups.

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APPENDIX A: BASES OF 2+1 LORENTZ GROUP REPRESENTATIONS AND S^μ MATRICES

Spin projection operators $\hat{S}^\mu$ acting in the space of the functions $f(x,\mathbf{z})$ of $x=(x^\mu)$ and two complex variables $z^1=z_2$, $z^2=-z_1$, $|z_1|^2-|z_2|^2=|z^2|-|z^1|=1$, have the form

$$\hat{S}^\mu=\frac{1}{2}(z\gamma^\mu\partial_z-\overset{*}{z}\overset{*}{\gamma}{}^\mu\partial_{\overset{*}{z}}),\quad z=(z^1\; z^2),\ \partial_z=(\partial/\partial z^1\;\partial/\partial z^2)^T, \tag{A1}$$

where $\gamma^\mu=(\sigma_3,i\sigma_2,-i\sigma_1)$. For $z=(z_1\; z_2)$ the relation $\hat{S}^\mu=-\frac{1}{2}(z\overset{*}{\gamma}{}^\mu\partial_z-\overset{*}{z}\gamma^\mu\partial_{\overset{*}{z}})$ is valid.

The polynomials of the power $2S$ in z, which correspond to finite-dimensional irreps T_S^0 of 2+1 Lorentz group, can be written in the form

$$T_S^0:\quad f_S(x,z)=\sum_{n=0}^{2S}\phi^n(z)\psi_n(x),\quad \phi^n(z)=(C_{2S}^n)^{1/2}\,(z^1)^{2S-n}(z^2)^n,\quad s^0=S-n, \tag{A2}$$

where s^0 is eigenvalue of $\hat{S}^0$, and C_{2S}^n are binomial coefficients. The quasipolynomials of the power $2S\le -1$, which correspond to infinite-dimensional unitary irreps $T_S^\pm$ of 2+1 Lorentz group, can be written in the form

$$T_S^+:\quad f_S(x,z)=\sum_{n=0}^{\infty}\phi^n(z)\psi_n(x),\quad \phi^n(z)=(C_{2S}^n)^{1/2}\,(z^2)^{2S-n}(z^1)^n,\quad s^0=-S+n,$$

$$T_S^-:\quad f_S(x,\overset{*}{z})=\sum_{n=0}^{\infty}\phi^n(z)\psi_n(x),\quad \phi^n(z)=(C_{2S}^n)^{1/2}\,(\overset{*}{z}{}^{\dot 2})^{2S-n}(\overset{*}{z}{}^{\dot 1})^n,\quad s^0=S-n, \tag{A3}$$

$$C_{2S}^n=\left(\frac{(-1)^n\Gamma(n-2S)}{n!\Gamma(-2S)}\right)^{1/2}.$$

There is a correspondence between the action of differential operators $\hat{S}^\mu$ on the functions $f(x,\mathbf{z}) = \phi(\mathbf{z})\psi(x)$ and the multiplication of matrices $\hat{S}^\mu$ by columns $\psi(x)$ composed of $\psi_n(x)$, $\hat{S}^\mu f(x,\mathbf{z}) = \phi(\mathbf{z})S^\mu\psi(x)$. For the finite-dimensional representations T_S^0 we have $(S^0)^\dagger = S^0$, $(S^k)^\dagger = -S^k$,

$$
\begin{aligned}
(S^0)_n{}^{n'} &= \delta_{nn'}(S-n), \qquad n = 0,1,\ldots,2S,\\
(S^1)_n{}^{n'} &= -\frac{i}{2}\left(\delta_{n\ n'+1}\sqrt{(2S-n+1)n} + \delta_{n+1\ n'}\sqrt{(2S-n)(n+1)}\right),\\
(S^2)_n{}^{n'} &= -\frac{1}{2}\left(\delta_{n\ n'+1}\sqrt{(2S-n+1)n} - \delta_{n+1\ n'}\sqrt{(2S-n)(n+1)}\right).
\end{aligned}
\tag{A4}
$$

Matrices S^μ satisfy the condition $(S^\mu)^\dagger = \Gamma S^\mu \Gamma$, where Γ is a diagonal matrix, $(\Gamma)_n{}^{n'} = (-1)^n\delta_{nn'}$. The substitution $z \to \overset{*}{z}$ in (A2) changes only signs of S^0 and S^2. For representations T_S^+ of discrete positive series is valid $(S^\mu)^\dagger = S^\mu$,

$$
\begin{aligned}
(S^0)_n{}^{n'} &= \delta_{nn'}(-S+n), \qquad n = 0,1,2,\ldots,\\
(S^1)_n{}^{n'} &= -\frac{1}{2}\left(\delta_{n\ n'+1}\sqrt{(n-1-2S)n} + \delta_{n+1\ n'}\sqrt{(n-2S)(n+1)}\right),\\
(S^2)_n{}^{n'} &= \frac{i}{2}\left(\delta_{n\ n'+1}\sqrt{(n-1-2S)n} - \delta_{n+1\ n'}\sqrt{(n-2S)(n+1)}\right).
\end{aligned}
\tag{A5}
$$

For T_S^- matrices S^1 have the same form, whereas S^0, S^2 change only their signs.

The case of representations of principal series, which is not bounded by the highest (lowest) weight, was considered in [30].

For the representations, which correspond to finite-dimensional irreps T_S^0, the decomposition (A2) can be written in terms of symmetric spin-tensors: $\psi_{\alpha_1\ldots\alpha_{2S}}(x) = \psi_{\alpha_{(1}\ldots\alpha_{2S)}}(x)$,

$$
f_S(x,z) = \psi_{\alpha_1\ldots\alpha_{2S}}(x) z^{\alpha_1}\ldots z^{\alpha_{2S}}. \tag{A6}
$$

Comparing the decompositions (A2) and (A6), we obtain the relation

$$
(C_{2S}^n)^{1/2}\psi_n(x) = \psi_{\underbrace{1\ldots1}_{2S-n}\underbrace{2\ldots2}_{n}}(x). \tag{A7}
$$

APPENDIX B: BASES OF 3+2 DE SITTER AND 3+1 LORENTZ GROUPS REPRESENTATIONS AND Γ^μ MATRICES

Consider polynomials of elements of Dirac z-spinor $Z_D = (z^\alpha, \overset{*}{\underline{z}}_{\dot\alpha})$. Any polynomial of power $2S$ can be decomposed in the basis of $(2S+3)!/(6(2S)!)$ monomials

$$
(z^1)^a(z^2)^b\overset{*}{\underline{z}}{}_{\dot 1}^{c}\overset{*}{\underline{z}}{}_{\dot 2}^{d}, \quad a+b+c+d = 2S.
$$

One may write out 16 operators, which conserve the power of polynomial:

$$\hat{S}^{\mu\nu} = \frac{1}{2}((\sigma^{\mu\nu})_{\alpha}{}^{\beta} z^{\alpha}\partial_{\beta} + (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \underline{\overset{*}{z}}_{\dot{\alpha}} \underline{\partial}^{\dot{\beta}}) - c.c., \tag{B1}$$

$$\hat{\Gamma}^{\mu} = \hat{V}^{\mu}_{12} + \hat{V}^{\mu}_{21} - c.c. = \frac{1}{2}\left(\bar{\sigma}^{\mu\dot{\alpha}\alpha} \underline{\overset{*}{z}}_{\dot{\alpha}} \partial_{\alpha} + \sigma^{\mu}{}_{\alpha\dot{\alpha}} z^{\alpha} \underline{\partial}^{\dot{\alpha}}\right) - c.c., \tag{B2}$$

$$\hat{\underline{\Gamma}}^{\mu} = i(\hat{V}^{\mu}_{12} - \hat{V}^{\mu}_{21}) + c.c. = \frac{i}{2}\left(\bar{\sigma}^{\mu\dot{\alpha}\alpha} \underline{\overset{*}{z}}_{\dot{\alpha}} \partial_{\alpha} - \sigma^{\mu}{}_{\alpha\dot{\alpha}} z^{\alpha} \underline{\partial}^{\dot{\alpha}}\right) + c.c., \tag{B3}$$

$$\hat{\Gamma}^{5} = \frac{1}{2}\left(z^{\alpha}\partial_{\alpha} - \underline{\overset{*}{z}}_{\dot{\alpha}} \underline{\partial}^{\dot{\alpha}}\right) + c.c., \tag{B4}$$

$$\hat{\mathcal{T}} = -\hat{S}^{R}_{3} = \frac{1}{2}\left(z^{\alpha}\partial_{\alpha} + \underline{\overset{*}{z}}_{\dot{\alpha}} \underline{\partial}^{\dot{\alpha}}\right) - c.c., \tag{B5}$$

where $\partial_{\alpha} = \partial/\partial z^{\alpha}$, $\underline{\partial}^{\dot{\alpha}} = \partial/\partial \underline{\overset{*}{z}}_{\dot{\alpha}}$,

$$(\sigma^{\mu\nu})_{\alpha}{}^{\beta} = -\frac{i}{4}(\sigma^{\mu}\bar{\sigma}^{\nu} - \sigma^{\nu}\bar{\sigma}^{\mu})_{\alpha}{}^{\beta}, \quad (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} = -\frac{i}{4}(\bar{\sigma}^{\mu}\sigma^{\nu} - \bar{\sigma}^{\nu}\sigma^{\mu})^{\dot{\alpha}}{}_{\dot{\beta}}, \tag{B6}$$

and *c.c.* is complex conjugate term corresponding the action in the space of polynomials of the elements of $\underline{Z}_D = (\underline{z}^{\alpha}, \overset{*}{z}_{\dot{\alpha}})$. Operator $\hat{\mathcal{T}}$ commutes with other 15 operators and defines $(\pm)$ power of the polynomials, for functions of Z_D and $\underline{Z}_D$ respectively. Operators (B1)-(B4) obey commutation relations of $so(3,3) \sim sl(4,R)$ algebra,

$$\begin{aligned} &[\hat{\Gamma}^5, \hat{S}^{\mu\nu}] = 0, \quad [\hat{\Gamma}^5, \hat{\Gamma}^{\mu}] = i\hat{\underline{\Gamma}}^{\mu}, \quad [\hat{\Gamma}^5, \hat{\underline{\Gamma}}^{\mu}] = -i\hat{\Gamma}^{\mu}, \\ &[\hat{\underline{\Gamma}}^{\mu}, \hat{\underline{\Gamma}}^{\nu}] = -i\hat{S}^{\mu\nu}, \quad [\hat{S}^{\lambda\mu}, \hat{\underline{\Gamma}}^{\nu}] = i(\eta^{\mu\nu}\hat{\underline{\Gamma}}^{\lambda} - \eta^{\lambda\nu}\hat{\underline{\Gamma}}^{\mu}), \quad [\hat{\Gamma}^{\mu}, \hat{\underline{\Gamma}}^{\nu}] = i\eta^{\mu\nu}\hat{\Gamma}^5, \end{aligned} \tag{B7}$$

see also (5.58),(5.59). Using the notations $\hat{S}^{4\mu} = \hat{\Gamma}^{\mu}$, $\hat{S}^{5\mu} = \hat{\underline{\Gamma}}^{\mu}$, $\hat{S}^{54} = \hat{\Gamma}^5$, one can rewrite commutation relations in the form (5.62), where $\eta_{55} = \eta_{44} = \eta_{00} = 1$, $\eta_{11} = \eta_{22} = \eta_{33} = -1$. However, for unitary representations of the Poincaré group all the generators and, in particular $\hat{B}^{R}_{3} = \mp i\hat{\Gamma}^5$ (for functions of Z_D and $\underline{Z}_D$ respectively), are Hermitian. Thus, setting $\hat{S}^{5\mu} = i\hat{\underline{\Gamma}}^{\mu}$, $\hat{S}^{54} = i\hat{\Gamma}^5$, for these representations it is natural to consider an algebra $so(4,2) \sim su(2,2)$ of Hermitian operators.

Supplementing generators $\hat{S}^{\mu\nu}$ of the Lorentz group by four operators $\hat{\Gamma}^{\mu}$ (or $\hat{\underline{\Gamma}}^{\mu}$), we obtain the algebra of the 3+2 de Sitter group $SO(3,2)$. Generators in finite-dimensional representations of $SO(3,2)$ obey the relations $\hat{\Gamma}^{0\dagger} = \hat{\Gamma}^0$, $\hat{\Gamma}^{k\dagger} = -\hat{\Gamma}^k$.

The linear functions of z $f(x,z) = Z_D\Psi_D(x)$ correspond to four-dimensional bispinor representation. In the space of columns $\Psi_D(x)$ the operators act as matrices

$$\hat{S}^{\mu\nu} \to \sigma^{\mu\nu}/2, \quad \hat{\Gamma}^{\mu} \to \gamma^{\mu}/2, \quad \hat{\Gamma}^5 \to \gamma^5/2, \quad \hat{\underline{\Gamma}}^{\mu} \to i\gamma^{\mu}\gamma^5/2, \quad \hat{\mathcal{T}} \to 1/2. \tag{B8}$$

In accordance with general theory, Dirac matrices and spin 1 Duffin–Kemmer matrices obey commutation relations of $so(3,3)$ algebra [130,131].

Using (2.61)-(2.63), we get for the action of the discrete transformations on the operators (B1)-(B5):

$$\begin{array}{cccccc} & \hat{S}^{\mu\nu} & \hat{\Gamma}^{\mu} & \hat{\underline{\Gamma}}^{\mu} & \hat{\Gamma}^5 & \hat{\mathcal{T}} \\ C & -1 & -1 & 1 & 1 & -1 \\ P, T' & (-1)^{\delta_{0\mu}+\delta_{0\nu}} & -(-1)^{\delta_{0\mu}} & (-1)^{\delta_{0\mu}} & -1 & 1 \\ T_{sch} & -(-1)^{\delta_{0\mu}+\delta_{0\nu}} & (-1)^{\delta_{0\mu}} & (-1)^{\delta_{0\mu}} & -1 & -1 \end{array} \tag{B9}$$

It is possible to construct two linear in $\hat{p}^\mu$ equations for the scalar functions $f(x, \mathbf{z})$, which are invariant under proper Poincaré group

$$(\hat{p}_\mu \hat{\Gamma}^\mu - \varkappa) f(x, \mathbf{z}) = 0, \qquad (\hat{p}_\mu \underline{\hat{\Gamma}}^\mu - \varkappa) f(x, \mathbf{z}) = 0, \tag{B10}$$

but in accordance with (B9) only operator $\hat{p}_\mu \hat{\Gamma}^\mu$ is invariant under space reflection, operator $\hat{p}_\mu \underline{\hat{\Gamma}}^\mu$ changes the sign. Thus only the first equation is invariant under space reflection.

Operators $\hat{\Gamma}^5$ and $\hat{p}_\mu \hat{\Gamma}^\mu$ commute with all the left generators of the Poincaré group but do not commute with each other, $[\hat{\Gamma}^5, \hat{p}_\mu \hat{\Gamma}^\mu] = i\hat{p}_\mu \underline{\hat{\Gamma}}^\mu$. Therefore chirality of massive particle describing by the equation $(p_\mu \hat{\Gamma}^\mu - ms) f(x, \mathbf{z}) = 0$ is uncertain. Operator $\hat{\mathcal{T}}$ commutes both with all left generators of the Poincaré group and with operators $\hat{\Gamma}^\mu$; therefore one may relate to this operator some conserved quantum number changing the sign under charge conjugation.

On the polynomials of four complex variables $z^\alpha, \overset{*}{\underline{z}}_{\dot{\alpha}}$ one can realize symmetric irreps $T_{[2S\,0\,0]}$ of $SL(4, R) \sim SO(3, 3)$. These irreps are a symmetric part of $2S$-times direct product of fundamental four-dimensional irreps $T_{[1\,0\,0]}$ and remain irreducible after the reduction on the subgroup $SO(3, 2)$, $T_{[2S\,0\,0]} \to T_{[2S\,0]}$. Notice that here we use the notation different from [56]: $[2S\, j]$ corresponds to $(j + S\, S)$ in the notation of [56].

We will consider two bases of finite-dimensional irrep $T_{[2s\,0]}$ of $SO(3, 2)$, namely, bases consisting of eigenfunctions of operators $\hat{\Gamma}^5$ or $\hat{\Gamma}^0$. The first basis corresponds to chiral representation,

$$\varphi^{m_1 m_2}_{j_1 j_2}(z, \overset{*}{\underline{z}}) = N^{1/2} (z^1)^{j_1 + m_1} (z^2)^{j_1 - m_1} \overset{*}{\underline{z}}{}_{\dot{1}}^{j_2 + m_2} \overset{*}{\underline{z}}{}_{\dot{2}}^{j_2 + m_2}, \tag{B11}$$

where $s = j_1 + j_2$, $\lambda = j_1 - j_2$, m_1 and m_2 are eigenvalues of the operators $\hat{M}_3$ and $\hat{N}_3$, which are the linear combinations of $\hat{S}_3$ and $\hat{B}_3$, see (5.13), $N = (2s)!/((j_1 + m_1)!(j_1 - m_1)!(j_2 + m_2)!(j_2 - m_2)!)$. Consisting of eigenfunctions of $\hat{\Gamma}^0$ basis

$$\phi^{n_1 n_2}_{k_1 k_2}(z, \overset{*}{\underline{z}}) = (N')^{1/2} (z^1 + \overset{*}{\underline{z}}_{\dot{1}})^{k_1 + n_1} (z^2 + \overset{*}{\underline{z}}_{\dot{2}})^{k_1 - n_1} (z^1 - \overset{*}{\underline{z}}_{\dot{1}})^{k_2 + n_2} (z^2 - \overset{*}{\underline{z}}_{\dot{2}})^{k_2 - n_2}, \tag{B12}$$

where $s = k_1 + k_2$, $N' = (2s)!/((k_1 + n_1)!(k_1 - n_1)!(k_2 + n_2)!(k_2 - n_2)!)$, for $s = 1/2$ corresponds to Dirac representation. The functions (B12) are eigenfunctions of operators $\hat{\Gamma}^0, \underline{\hat{\Gamma}}^3, \hat{S}_3$ with eigenvalues $k_1 - k_2$, $i(n_1 - n_2)/2$, $(n_1 + n_2)/2$ respectively. For fixed s we have

$$f_s(x, z, \overset{*}{\underline{z}}) = \sum_{j_1 + j_2 = s} \sum_{m_1, m_2} \psi^{m_1 m_2}_{j_1 j_2}(x) \varphi^{m_1 m_2}_{j_1 j_2}(z, \overset{*}{\underline{z}}) = \sum_{k_1 + k_2 = s} \sum_{n_1, n_2} \psi^{n_1 n_2}_{k_1 k_2}(x) \phi^{n_1 n_2}_{k_1 k_2}(z, \overset{*}{\underline{z}}). \tag{B13}$$

Below we will use the basis (B12). According to (5.74) in the rest frame for a particle describing by the system (5.71)-(5.73) we have

$$\begin{aligned} &f(x, z, \overset{*}{\underline{z}}) = \psi^+(x) \phi^+_{s,s^3}(z, \overset{*}{\underline{z}}) + \psi^-(x) \phi^-_{s,s^3}(z, \overset{*}{\underline{z}}) = C_1 e^{imx^0} \phi^+_{s,s^3}(z, \overset{*}{\underline{z}}) + C_2 e^{-imx^0} \phi^-_{s,s^3}(z, \overset{*}{\underline{z}}), \\ &\phi^+_{s,s^3}(z, \overset{*}{\underline{z}}) = (z^1 + \overset{*}{\underline{z}}_{\dot{1}})^{s + s^3} (z^2 + \overset{*}{\underline{z}}_{\dot{2}})^{s - s^3}, \quad \phi^-_{s,s^3}(z, \overset{*}{\underline{z}}) = (z^1 - \overset{*}{\underline{z}}_{\dot{1}})^{s + s^3} (z^2 - \overset{*}{\underline{z}}_{\dot{2}})^{s - s^3}. \end{aligned} \tag{B14}$$

The equation $(\hat{p}_\mu \hat{\Gamma}^\mu - sm) f(x, z, \overset{*}{\underline{z}}) = 0$ has the matrix form

$$(\hat{p}_\mu \Gamma^\mu - sm) \psi(x) = 0, \tag{B15}$$

where $\psi(x)$ is a column. It is convenient to enumerate the basis elements (B12) (and the elements of the column $\psi(x)$) in order of decrease of $k_1 - k_2 = s, s-1, \ldots, -s$. Matrices Γ^μ obey the relations $\Gamma^{0\dagger} = \Gamma^0$, $\Gamma^{k\dagger} = -\Gamma^k$. Matrix Γ^0 is diagonal and has the elements $k_1 - k_2$. Matrices Γ^1 and Γ^3 are skew-symmetric real, and Γ^2 is symmetric imaginary. According to (B2) matrices Γ^k have nonzero elements only in blocks corresponding to the transitions $(k_1, k_2) \to (k_1 \pm 1/2, k_2 \mp 1/2)$. Using this property, it is easy to see that diagonal matrix Γ with the elements $(-1)^{2k_2}$ commutes with Γ^0 and anticommutes with Γ^k, $\Gamma^{\mu\dagger} = \Gamma\Gamma^\mu\Gamma$. This allows one to rewrite the Hermitian-conjugate equation $\psi^\dagger(\overleftarrow{\hat{p}_\mu}\Gamma^{\mu\dagger} + sm) = 0$ in the form

$$\bar{\psi}(x)(\overleftarrow{\hat{p}_\mu}\Gamma^\mu + sm) = 0, \quad \bar{\psi} = \psi^\dagger\Gamma, \tag{B16}$$

and to define invariant scalar product in the space of columns as $\int \bar{\psi}(x)\psi(x)d^3x$. As a consequence of (B15) and (B16), the continuity equation holds

$$\partial_\mu j^\mu = 0, \quad j^\mu = \bar{\psi}\Gamma^\mu\psi.$$

Now the question concerning the positive definiteness of current vector component j^0 and energy density may be consider similarly to 2+1-dimensional case, see Section 3. For half-integer spin particles describing by the system (5.71)-(5.73) charge density j^0 is positive definite, since in the rest frame (see (B14)) $j^0 = \bar{\psi}\Gamma\Gamma^0\psi = s(|\psi^+(x)| + |\psi^-(x)|) > 0$. Energy density (defined in terms of energy-momentum tensor (4.27)) and the scalar product $\bar{\psi}\psi$ are indefinite since in the rest frame they are proportional to $|\psi^+(x)| - |\psi^-(x)|$. For integer spin particles energy density is positive definite, the scalar product and j^0 are indefinite.

Consider discrete transformations in terms of the columns $\psi(x)$. According to (5.8) under space reflection $\phi^{n_1n_2}_{k_1k_2}(z, \overset{*}{\underline{z}}) \to (-1)^{2k_1}\phi^{n_1n_2}_{k_1k_2}(z, \overset{*}{\underline{z}})$. Whence, taking into account $f(x, z, \overset{*}{\underline{z}}) \to f(x', z') = \phi(z, \overset{*}{\underline{z}})\psi'(x')$, we get

$$\psi(x) \xrightarrow{P} (-1)^{2s}\Gamma\psi(\bar{x}), \quad \text{where } \bar{x} = (x^0, -x^k). \tag{B17}$$

According to (2.63) under charge conjugation $\phi^{n_1n_2}_{k_1k_2}(z, \overset{*}{\underline{z}}) \to \phi^{n_1n_2}_{k_1k_2}(\overset{*}{z}, \underline{z})$. Taking into account that $\phi^{n_1n_2}_{k_1k_2}(\overset{*}{z}, \underline{z})$ and $(-1)^{s+n_1-n_2}\phi^{n_2n_1}_{k_2k_1}(z, \overset{*}{\underline{z}})$ have the same transformation rule, we get

$$\psi^{n_1n_2}_{k_1k_2}(x) \xrightarrow{C} (-1)^{s+n_1-n_2}\overset{*}{\psi}{}^{n_2\,n_1}_{k_2\,k_1}(x). \tag{B18}$$

In particular, for $s = 1/2$, using the relation $f(x, z, \overset{*}{\underline{z}}) = Z_D\Psi(x)$, we get $\Psi(x) \xrightarrow{P} \gamma^0\Psi(\bar{x})$, $\Psi(x) \xrightarrow{C} \Psi^c(x) = C\bar{\Psi}^T(x)$, where C is the matrix with elements $-i\sigma_2$ on secondary diagonal, $C = i\gamma^2\gamma^0$. The transformation properties of the bilinears $\bar{\psi}\Gamma^\mu\psi$, $\bar{\psi}\Gamma^5\psi$, $\bar{\psi}\underline{\Gamma}^\mu\psi$ under C, P, T coincide with ones of the corresponding operators, see (B9).

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